

# Recent results about the detection of unknown boundaries and inclusions in elastic plates \*

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**Abstract.** In this paper we review some recent results concerning inverse problems for thin elastic plates. The plate is assumed to be made by non-homogeneous linearly elastic material belonging to a general class of anisotropy. A first group of results concerns uniqueness and stability for the determination of unknown boundaries, including the cases of cavities and rigid inclusions. In the second group of results, we consider upper and lower estimates of the area of unknown inclusions given in terms of the work exerted by a couple field applied at the boundary of the plate. In particular, we extend previous size estimates for elastic inclusions to the case of cavities and rigid inclusions.

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## 1 Introduction

The problems we consider in the present paper belong to a more general issue that has evolved in the last fifteen years in the field of inverse problems. Such an issue collects the problems of determining, by a finite number of boundary measurements, unknown boundaries and inclusions entering the boundary value problems for partial differential equations and systems of

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elliptic and parabolic type. Such problems arise in nondestructive techniques by electrostatic measurements [Ka-Sa], [In], in thermal imaging [Vo-Mo], [Br-C], in elasticity theory [N], [Bo-Co], [M-R3], and in many other similar applications [Is2]. In this paper we try to enlighten the different facets of the issue fixing our attention on the theory of thin elastic plates. In Section 3 we give a self contained derivation of the Kirchhoff-Love plate model on which such a theory is based.

We begin with the problem of the determination of a rigid inclusion embedded in a thin elastic plate.

Let  $\Omega$  denote the middle plane of the plate. We assume that  $\Omega$  is a bounded domain of  $\mathbb{R}^2$  of class  $C^{1,1}$ . Let  $h$  be its constant thickness,  $h \ll \text{diam}(\Omega)$ . The rigid inclusion  $D$  is modeled as an open simply connected domain compactly contained in  $\Omega$ , with boundary of class  $C^{1,1}$ . The transversal displacement  $w \in H^2(\Omega)$  of the plate satisfies the following mixed boundary value problem, see, for example, [Fi] and [Gu],

$$\left\{ \begin{array}{ll} \text{div}(\text{div}(\mathbb{P}\nabla^2 w)) = 0, & \text{in } \Omega \setminus \overline{D}, \\ (\mathbb{P}\nabla^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \text{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega, \\ w|_{\overline{D}} \in \mathcal{A}, & \text{in } \overline{D}, \\ w_{,n}^e = w_{,n}^i, & \text{on } \partial D, \end{array} \right. \quad \begin{array}{l} (1.1) \\ (1.2) \\ (1.3) \\ (1.4) \\ (1.5) \end{array}$$

coupled with the *equilibrium conditions* for the rigid inclusion  $D$

$$\int_{\partial D} (\text{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s}) g - ((\mathbb{P}\nabla^2 w)n \cdot n)g_{,n} = 0, \quad \text{for every } g \in \mathcal{A}, \quad (1.6)$$

where  $\mathcal{A}$  denotes the space of affine functions. In the above equations,  $n$  and  $\tau$  are the unit outer normal and the unit tangent vector to  $\Omega \setminus \overline{D}$ , respectively, and we have defined  $w^e \equiv w|_{\Omega \setminus \overline{D}}$  and  $w^i \equiv w|_{\overline{D}}$ . Moreover,  $\widehat{M}_\tau$ ,  $\widehat{M}_n$  are the twisting and bending components of the assigned couple field  $\widehat{M}$ , respectively. The plate tensor  $\mathbb{P}$  is given by  $\mathbb{P} = \frac{h^3}{12}\mathbb{C}$ , where  $\mathbb{C}$  is the elasticity tensor describing the response of the material of the plate. We assume that  $\mathbb{C}$  has cartesian components  $C_{ijkl}$ ,  $i, j, k, l = 1, 2$ , which satisfy the standard symmetry conditions (4.2), the regularity assumption (4.3) and the strong convexity condition (4.4).

Given any  $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$ , satisfying the compatibility conditions  $\int_{\partial\Omega} \widehat{M}_i = 0$ , for  $i = 1, 2$ , problem (1.1)–(1.6) admits a solution  $w \in H^2(\Omega)$ , which is uniquely determined up to addition of an affine function.

Let us denote by  $\Gamma$  an open portion within  $\partial\Omega$  representing the part of the boundary where measurements are taken.

The inverse problem consists in determining  $D$  from the measurement of  $w$  and  $w_{,n}$  on  $\Gamma$ . For instance, the uniqueness issue can be formulated as follows: *Given two solutions  $w_i$  to (1.1)–(1.6) for  $D = D_i$ ,  $i = 1, 2$ , satisfying*

$$w_1 = w_2, \text{ on } \Gamma, \quad (1.7)$$

$$w_{1,n} = w_{2,n}, \text{ on } \Gamma, \quad (1.8)$$

*does  $D_1 = D_2$  hold?*

It is convenient to replace each solution  $w_i$  introduced above with  $v_i = w_i - g_i$ , where  $g_i$  is the affine function which coincides with  $w_i$  on  $\partial D_i$ ,  $i = 1, 2$ . By this approach, maintaining the same letter to denote the solution, we rephrase the equilibrium problem (1.1)–(1.5) in terms of the following mixed boundary value problem with homogeneous Dirichlet conditions on the boundary of the rigid inclusion

$$\left\{ \begin{array}{ll} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w)) = 0, & \text{in } \Omega \setminus \overline{D}, \quad (1.9) \\ (\mathbb{P}\nabla^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \quad (1.10) \\ \operatorname{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega, \quad (1.11) \\ w = 0, & \text{on } \partial D, \quad (1.12) \\ \frac{\partial w}{\partial n} = 0, & \text{on } \partial D, \quad (1.13) \end{array} \right.$$

coupled with the *equilibrium conditions* (1.6), which has a unique solution  $w \in H^2(\Omega \setminus \overline{D})$ . Therefore, the uniqueness question may be rephrased as follows:

*Given two solutions  $w_i$  to (1.9)–(1.13), (1.6) for  $D = D_i$ ,  $i = 1, 2$ , satisfying, for some  $g \in \mathcal{A}$*

$$w_1 - w_2 = g, \quad (w_1 - w_2)_{,n} = g_{,n}, \quad \text{on } \Gamma, \quad (1.14)$$

*does  $D_1 = D_2$  hold?*

Obviously, the inverse problem above is equivalent to the determination of the portion  $\partial D$  of the boundary of  $\Omega \setminus \overline{D}$  in the boundary value problem (1.9)–(1.13).

In the stability issue, instead of (1.7) and (1.8), we assume

$$\min_{g \in \mathcal{A}} \left\{ \|w_1 - w_2 - g\|_{L^2(\Gamma)} + \|(w_1 - w_2 - g)_{,n}\|_{L^2(\Gamma)} \right\} \leq \epsilon, \quad (1.15)$$

for some  $\epsilon > 0$ , and we ask for the following estimate

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \eta(\epsilon), \quad (1.16)$$

where  $\eta(\epsilon)$  is a suitable infinitesimal function.

The uniqueness for the problem above has been proved in [M-R3] under the a priori assumption of  $C^{3,1}$  regularity of  $\partial D$  and with only one nontrivial couple field. Here, by nontrivial we mean that

$$(\widehat{M}_n, \widehat{M}_{\tau,s}) \neq 0. \quad (1.17)$$

Concerning the stability issue, in [M-R-Ve5] we have proved a log-log type estimate, namely in inequality (1.16) we have  $\eta(\epsilon) = O((\log |\log \epsilon|)^{-\alpha})$ , where the positive parameter  $\alpha$  depends on the a priori data, see Theorem 4.3 below for a precise statement.

In [M-R-Ve2] the inverse problem of determining a cavity in an elastic plate has been faced. We recall that in such a case conditions (1.4)-(1.5) are replaced by homogeneous Neumann boundary conditions, which are much more difficult to handle with respect to Dirichlet boundary conditions arising in the case of rigid inclusions. For this reason a uniqueness result has been established making *two* linearly independent boundary measurements. In [M-R3] it has also been proved a uniqueness result for a variant of the problems considered above, that is the case of a plate whose boundary has an unknown and inaccessible portion where  $\widehat{M} = 0$ . In this case, thanks to the more favorable geometric situation, one measurement suffices to detect the unknown boundary portion. The corresponding stability results for these two cases have not yet been proved.

The methods used to prove the above mentioned uniqueness and stability results are based on unique continuation properties and quantitative estimates of unique continuation for solutions to the plate equation (1.1). Since such properties and estimates are consequences of the three sphere inequality for solutions to equation (1.1), we will discuss a while about the main features of such inequality.

The three sphere inequality for solutions to partial differential equations and systems has a long and interesting history that intertwines with the issue of unique continuation properties and the issue of stability estimates [Al-M], [Ho85], [Is1], [Jo], [Lan], [Lav], [Lav-Rom-S], [L-N-W], [L-Nak-W]. In many important cases, the three sphere inequality is the elementary tool to prove various types of quantitative estimates of unique continuation such as, for example, stability estimates for the Cauchy problem, smallness propagation estimates and quantitative evaluation of the vanishing rate of solutions to PDEs. Such questions have been intensively studied in the context of second order equations of elliptic and parabolic type. We refer to [Al-Ro-R-Ve] and [Ve2] where these topics are widely investigated for such types of equations.

The three sphere inequality for equation (1.1) has been proved in [M-R-Ve5] under the very general assumption that the elastic material of the plate is

anisotropic and obeys the so called *dichotomy condition*. Roughly speaking, such a condition implies that the plate operator at the left hand side of (1.1) can be written as  $L_2 L_1 + Q$ , where  $L_2, L_1$  are second order elliptic operators with  $C^{1,1}$  coefficients and  $Q$  is a third order operator with bounded coefficients. For more details we refer to (4.5a)-(4.5b) below and [M-R-Ve5]. A simplified version of such inequality is the following one

$$\int_{B_{r_2}(x_0)} |\nabla^2 w|^2 \leq C \left( \int_{B_{r_1}(x_0)} |\nabla^2 w|^2 \right)^\delta \left( \int_{B_{r_3}(x_0)} |\nabla^2 w|^2 \right)^{1-\delta}, \quad (1.18)$$

for every  $r_1 < r_2 < r_3$ , where  $\delta \in (0, 1)$  and  $C$  depend only on the parameters related to the regularity, ellipticity and dichotomy conditions assumed on  $\mathbb{C}$ , and on the ratios  $r_1/r_2, r_2/r_3$ ; in particular,  $\delta$  and  $C$  *do not depend* on  $w$ .

Previously, the three sphere inequality was proved in [M-R-Ve1], for the isotropic plate (that is  $C_{ijkl}(x) = \delta_{ij}\delta_{kl}\lambda(x) + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mu(x)$ ,  $i, j, k, l = 1, 2$ ) and in [Ge] for the class of fourth (and higher) order elliptic equation  $\mathcal{L}u = 0$  where  $\mathcal{L} = L_2 L_1$  and  $L_2, L_1$  are second order elliptic equation with  $C^{1,1}$  coefficients.

The proof of the stability result, of which we give a sketch in Subsection 4.2, has essentially the same structure of the proofs of analogous stability results in the following context:

a) Second order elliptic equations: [Al-Ro], [Be-Ve], [Ro] (two variables elliptic equations); [Al-B-R-Ve1], [Al-B-R-Ve2], [Si] (several variables elliptic equations)

b) Second order parabolic equation: [C-R-Ve1], [C-R-Ve2], [Dc-Ro-Ve], [B-Dc-Si-Ve], [Ve1], [Ve2]

c) Elliptic systems: [M-R3], [M-R4] (elasticity); [Ba] (Stokes fluid)

It is important to say that the stability estimates proved in the papers of list a) and b) are of logarithmic type, that is an optimal rate of convergence, as shown by counterexamples ([Al1], [Dc-Ro] for case a) and [Dc-Ro-Ve], [Ve2] for case b)). The stability estimates proved in the papers of list c) and in the case of plate equation are of log-log type, that is with a worse rate of convergence. It seems difficult to improve such an estimate. We believe that the main difficulty to get such an improvement is due to the lack of quantitative estimates of *strong unique continuation property at the boundary*. In order to give an idea of the crucial point which marks the difference between these cases, let us notice that, by iterated application of the three sphere inequality (1.18) it can be proved there exists  $\bar{\rho} > 0$  such that for every  $\rho \in (0, \bar{\rho})$  and every  $\bar{x} \in \partial D_j$ ,  $j = 1, 2$ , the following inequality holds true

$$\int_{B_\rho(\bar{x}) \cap (\Omega \setminus \overline{D_j})} |\nabla^2 w_j|^2 \geq C \exp(-A\rho^{-B}), \quad (1.19)$$

where  $A > 0$ ,  $B > 0$  and  $C > 0$  only depend on the a priori information, in particular they depend by the quantity (frequency)

$$\frac{\|\widehat{M}\|_{L^2(\partial\Omega, \mathbb{R}^2)}}{\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}}. \quad (1.20)$$

In cases a) and b), it is possible to prove a refined form of inequality (1.19), in which the exponential term is replaced with a positive power of  $\rho$ , obtaining a quantitative estimate of strong unique continuation property at the boundary. It has been shown in [Al-B-R-Ve1] that this is a key ingredient in proving that the stability estimate for the corresponding inverse problem with unknown boundaries in the conductivity context is not worse than logarithm. This mathematical tool is available for second order elliptic, [A-E], and parabolic equations, [Es-Fe-Ve], but is not currently available for elliptic systems and plate equation. This happens even in the simplest case of isotropic material, and this is the reason for the presence of a double logarithm in our stability estimate. Finally, as remarked in [M-R-Ve4], it seems hopeless the possibility that solutions to (1.1) can satisfy even a strong unique continuation property in the interior, without any a priori assumption on the anisotropy of the material, see also [Ali]. Regarding this point, our dichotomy condition (4.5a)-(4.5b) basically contains the same assumptions under which the unique continuation property holds for a fourth order elliptic equation in two variables.

In the present paper we have also proved constructive upper and lower estimates of the area of a rigid inclusion or of a cavity,  $D$ , in terms of an easily expressed quantity related to work. More precisely, suppose we make the following diagnostic test. We take a reference plate, i.e. a plate without inclusion or cavity, and we deform it by applying a couple field  $\widehat{M}$  at the boundary  $\partial\Omega$ . Let  $W_0$  be the work exerted in deforming the specimen. Next, we repeat the same experiment on a possibly defective plate. The exerted work generally changes and assumes, say, the value  $W$ . We are interested in finding constructive estimates, from above and from below, of the *area* of  $D$  in terms of the difference  $|W - W_0|$ . In order to prove such estimates we proceed along the path outlined in [M-R-Ve1] and [M-R-Ve6] in which the inclusion inside the plate is made by different elastic material. In this introduction we illustrate such *intermediate* case, since the scheme of the mathematical procedure is fairly simple to describe. With regard to this intermediate case we also want to stress that, in contrast to the extreme cases, there are not available any kind of uniqueness result for the inverse problem of determining inclusion  $D$  from the knowledge of a finite number of measurements on the boundary. This appears to be an extremely difficult

problem. In fact, despite the wide research developed in this field, a general uniqueness result has not been obtained yet even in the simpler context arising in electrical impedance tomography (which involves a second order elliptic equation), see, for instance, [Is1] and [Al1] for an extensive reference list.

Denoting, as above, by  $w$  the transversal displacement of the plate and by  $\widehat{M}_\tau, \widehat{M}_n$  the twisting and bending components of the assigned couple field  $\widehat{M}$ , respectively, the infinitesimal deformation of the defective plate is governed by the fourth order Neumann boundary value problem

$$\begin{cases} \operatorname{div}(\operatorname{div}((\chi_{\Omega \setminus D}\mathbb{P} + \chi_D\widetilde{\mathbb{P}})\nabla^2 w)) = 0, & \text{in } \Omega, \\ (\mathbb{P}\nabla^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \operatorname{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega. \end{cases} \quad (1.21)$$

$$\begin{cases} (\mathbb{P}\nabla^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \end{cases} \quad (1.22)$$

$$\begin{cases} \operatorname{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega. \end{cases} \quad (1.23)$$

In the above equations,  $\chi_D$  denotes the characteristic function of  $D$ . The plate tensors  $\mathbb{P}, \widetilde{\mathbb{P}}$  are given by

$$\mathbb{P} = \frac{h^3}{12}\mathbb{C}, \quad \widetilde{\mathbb{P}} = \frac{h^3}{12}\widetilde{\mathbb{C}}, \quad (1.24)$$

where  $\mathbb{C}$  is the elasticity tensor describing the response of the material in the reference plate  $\Omega$  and satisfies the usual symmetry conditions (4.2), regularity condition (4.3), strong convexity condition (4.4) and the *dichotomy condition*, whereas  $\widetilde{\mathbb{C}}$  denotes the (unknown) corresponding tensor for the inclusion  $D$ .

The work exerted by the couple field  $\widehat{M}$  has the expression

$$W = - \int_{\partial\Omega} \widehat{M}_{\tau,s} w + \widehat{M}_n w_{,n}. \quad (1.25)$$

When the inclusion  $D$  is absent, the equilibrium problem (1.21)-(1.23) becomes

$$\begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w_0)) = 0, & \text{in } \Omega, \\ (\mathbb{P}\nabla^2 w_0)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \operatorname{div}(\mathbb{P}\nabla^2 w_0) \cdot n + ((\mathbb{P}\nabla^2 w_0)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega, \end{cases} \quad (1.26)$$

$$\begin{cases} (\mathbb{P}\nabla^2 w_0)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \end{cases} \quad (1.27)$$

$$\begin{cases} \operatorname{div}(\mathbb{P}\nabla^2 w_0) \cdot n + ((\mathbb{P}\nabla^2 w_0)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega, \end{cases} \quad (1.28)$$

where  $w_0$  is the transversal displacement of the reference plate. The corresponding work exerted by  $\widehat{M}$  is given by

$$W_0 = - \int_{\partial\Omega} \widehat{M}_{\tau,s} w_0 + \widehat{M}_n w_{0,n}. \quad (1.29)$$

In [M-R-Ve6] the following result has been proved. Assuming that the following *fatness-condition* is satisfied

$$\text{area}(\{x \in D \mid \text{dist}\{x, \partial D\} > h_1\}) \geq \frac{1}{2} \text{area}(D), \quad (1.30)$$

where  $h_1$  is a given positive number, then

$$C_1 \left| \frac{W - W_0}{W_0} \right| \leq \text{area}(D) \leq C_2 \left| \frac{W - W_0}{W_0} \right|, \quad (1.31)$$

where the constants  $C_1, C_2$  only depend on the a priori data. Besides the assumptions on the plate tensor  $\mathbb{C}$  given above, estimates (1.31) are established under some suitable assumption on the jump  $\tilde{\mathbb{C}} - \mathbb{C}$ .

In the *extreme* cases corresponding to a rigid inclusion or a cavity  $D$ , the estimates are a little more involute than (1.31) and additional regularity conditions on the boundary  $\partial D$  and on the plate tensor  $\mathbb{P}$  are generally required. The main difference between *extreme* and *intermediate* cases lies in the estimate from below of  $|D|$ . Indeed, in the former case we use regularity estimates (in the interior) for the reference solution  $w_0$  to equation (1.26), whereas in the latter we combine regularity estimates, trace and Poincaré inequalities. On the other hand, the argument used for the estimate from above of  $|D|$  is essentially the same as in [M-R-Ve6] and involves quantitative estimates of unique continuation in the form of three sphere inequality for the hessian  $\nabla^2 w_0$ . It is exactly at this point that the dichotomy condition (4.5a)–(4.5b) on the tensor  $\mathbb{C}$  is needed.

The analogous bounds in plate theory for *intermediate* inclusions were first obtained when the reference plate satisfies isotropic conditions, [M-R-Ve1], extended to anisotropic materials satisfying the dichotomy conditions, [M-R-Ve6], obtained in a weaker form for general inclusions in absence of the fatness condition, [M-R-Ve3], and recently in the context of shallow shells in [Dc-Li-Wa] and [Dc-Li-Ve-Wa]. The reader is referred to [K-S-Sh], [Al-R], [Al-R-S], [Be-Fr-Ve] for size estimates of inclusions in the context of the electrical impedance tomography and to [Ik], [Al-M-R2], [Al-M-R3], [Al-M-R-V] for corresponding problems in two and three-dimensional linear elasticity. See also [L-Dc-N] for an application of the size estimates approach in thermography.

Size estimates for *extreme* inclusions were obtained in [Al-M-R1] for electric conductors and in [M-R1] for elastic bodies, see also [Al-M-R2].

The paper is organized as follows. In Section 2 we collect some notation. In Section 3 we provide a derivation of the Kirchhoff-Love model of the plate. In Section 4 we present the uniqueness and stability results concerning the



determination of rigid inclusions, and the uniqueness results for the case of cavities and unknown boundary portions. In particular we have focused our attention on the case of rigid inclusions and, for a better comprehension of the arguments, we have recalled the proof of the uniqueness result, using it as a base for a sketch of the more complex proof of the stability result. Section 5 contains the estimates of the area of *extreme* inclusions.

## 2 Notation

Let  $P = (x_1(P), x_2(P))$  be a point of  $\mathbb{R}^2$ . We shall denote by  $B_r(P)$  the disk in  $\mathbb{R}^2$  of radius  $r$  and center  $P$  and by  $R_{a,b}(P)$  the rectangle  $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a, |x_2 - x_2(P)| < b\}$ . To simplify the notation, we shall denote  $B_r = B_r(O)$ ,  $R_{a,b} = R_{a,b}(O)$ .

**Definition 2.1.** ( $C^{k,1}$  regularity) Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ . Given  $k \in \mathbb{N}$ , we say that a portion  $S$  of  $\partial\Omega$  is of *class  $C^{k,1}$  with constants  $\rho_0, M_0 > 0$* , if, for any  $P \in S$ , there exists a rigid transformation of coordinates under which we have  $P = 0$  and

$$\Omega \cap R_{\frac{\rho_0}{M_0}, \rho_0} = \{x = (x_1, x_2) \in R_{\frac{\rho_0}{M_0}, \rho_0} \mid x_2 > \psi(x_1)\},$$

where  $\psi$  is a  $C^{k,1}$  function on  $\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)$  satisfying

$$\psi(0) = 0, \quad \psi'(0) = 0, \quad \text{when } k \geq 1,$$

$$\|\psi\|_{C^{k,1}\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)} \leq M_0 \rho_0.$$

When  $k = 0$  we also say that  $S$  is of *Lipschitz class with constants  $\rho_0, M_0$* .

*Remark 2.2.* We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous with the  $L^\infty$  norm and coincide with the standard definition when the dimensional parameter equals one, see [M-R-Ve5] for details.

Given a bounded domain  $\Omega$  in  $\mathbb{R}^2$  such that  $\partial\Omega$  is of class  $C^{k,1}$ , with  $k \geq 1$ , we consider as positive the orientation of the boundary induced by the outer unit normal  $n$  in the following sense. Given a point  $P \in \partial\Omega$ , let us denote by  $\tau = \tau(P)$  the unit tangent at the boundary in  $P$  obtained by applying to  $n$  a counterclockwise rotation of angle  $\frac{\pi}{2}$ , that is  $\tau = e_3 \times n$ , where  $\times$  denotes the vector product in  $\mathbb{R}^3$ ,  $\{e_1, e_2\}$  is the canonical basis in  $\mathbb{R}^2$  and  $e_3 = e_1 \times e_2$ . Given any connected component  $\mathcal{C}$  of  $\partial\Omega$  and fixed

a point  $P \in \mathcal{C}$ , let us define as positive the orientation of  $\mathcal{C}$  associated to an arclength parametrization  $\varphi(s) = (x_1(s), x_2(s))$ ,  $s \in [0, l(\mathcal{C})]$ , such that  $\varphi(0) = P$  and  $\varphi'(s) = \tau(\varphi(s))$ , where  $l(\mathcal{C})$  denotes the length of  $\mathcal{C}$ .

Throughout the paper, we denote by  $\partial_i u$ ,  $\partial_s u$ , and  $\partial_n u$  the derivatives of a function  $u$  with respect to the  $x_i$  variable, to the arclength  $s$  and to the normal direction  $n$ , respectively, and similarly for higher order derivatives.

We denote by  $\mathbb{M}^2$  the space of  $2 \times 2$  real valued matrices and by  $\mathcal{L}(X, Y)$  the space of bounded linear operators between Banach spaces  $X$  and  $Y$ .

For every  $2 \times 2$  matrices  $A, B$  and for every  $\mathbb{L} \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$ , we use the following notation:

$$(\mathbb{L}A)_{ij} = L_{ijkl}A_{kl}, \quad (2.1)$$

$$A \cdot B = A_{ij}B_{ij}, \quad |A| = (A \cdot A)^{\frac{1}{2}}. \quad (2.2)$$

Notice that here and in the sequel summation over repeated indexes is implied.

Finally, let us introduce the linear space of the affine functions on  $\mathbb{R}^2$

$$\mathcal{A} = \{g(x_1, x_2) = ax_1 + bx_2 + c, \ a, b, c \in \mathbb{R}\}.$$

### 3 The Kirchhoff-Love plate model

In the last two decades different methods were used to provide new justification of the theory of thin plates. Among these, we recall the method of asymptotic expansion [C-D], the method of internal constraints [PG], [L-PG], the theory of  $\Gamma$ -convergence in conjunction with appropriate averages [A-B-P] or on rescaled domain and with rescaled displacements [B-C-G-R], [Pa1], and weak convergence methods on a rescaled domain and with rescaled displacements [C]. We refer the interested reader to [Pa2] for a recent account of the advanced results on this topic. The present section has a more modest aim: to show how to deduce the equations governing the statical equilibrium of an elastic thin plate following the classical approach of the Theory of Structures.

Let us consider a thin plate  $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$  with middle surface represented by a bounded domain  $\Omega$  in  $\mathbb{R}^2$  having uniform thickness  $h$ ,  $h \ll \text{diam}(\Omega)$ , and boundary  $\partial\Omega$  of class  $C^{1,1}$ . Only in this section, we adopt the convention that Greek indexes assume the values 1, 2, whereas Latin indexes run from 1 to 3.

We follow the direct approach to define the infinitesimal deformation of the plate. In particular, we restrict ourselves to the case in which the points  $x = (x_1, x_2)$  of the middle surface  $\Omega$  are subject to transversal displacement  $w(x_1, x_2)e_3$ , and any transversal material fiber  $\{x\} \times [-\frac{h}{2}, \frac{h}{2}]$ ,  $x \in \Omega$ ,

undergoes an infinitesimal rigid rotation  $\omega(x)$ , with  $\omega(x) \cdot e_3 = 0$ . In this section we shall be concerned exclusively with regular functions on their domain of definition. For example, the above functions  $w$  and  $\omega$  are such that  $w \in C^\infty(\overline{\Omega}, \mathbb{R})$  and  $\omega \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$ . These conditions are unnecessarily restrictive, but this choice simplifies the mechanical formulation of the equilibrium problem. The above kinematical assumptions imply that the displacement field present in the plate is given by the following three-dimensional vector field:

$$u(x, x_3) = w(x)e_3 + x_3\varphi(x), \quad x \in \overline{\Omega}, \quad |x_3| \leq \frac{h}{2}, \quad (3.1)$$

where

$$\varphi(x) = \omega(x) \times e_3, \quad x \in \overline{\Omega}. \quad (3.2)$$

By (3.1) and (3.2), the associated infinitesimal strain tensor  $E[u] \in \mathbb{M}^3$  takes the form

$$E[u](x, x_3) \equiv (\nabla u)^{sym}(x, x_3) = x_3(\nabla_x \varphi(x))^{sym} + (\gamma(x) \otimes e_3)^{sym}, \quad (3.3)$$

where  $\nabla_x(\cdot) = \frac{\partial}{\partial x_\alpha}(\cdot)e_\alpha$  is the surface gradient operator,  $\nabla^{sym}(\cdot) = \frac{1}{2}(\nabla(\cdot) + \nabla^T(\cdot))$ , and

$$\gamma(x) = \varphi(x) + \nabla_x w(x). \quad (3.4)$$

Within the approximation of the theory of infinitesimal deformations,  $\gamma$  is the angular deviation between the transversal material fiber at  $x$  and the normal direction to the deformed middle surface of the plate at  $x$ . In Kirchhoff-Love theory it is assumed that every transversal material fiber remains normal to the deformed middle surface, e.g.  $\gamma = 0$  in  $\Omega$ .

The traditional deduction of the mechanical model of a thin plate follows essentially from integration over the thickness of the corresponding three-dimensional quantities. In particular, taking advantage of the infinitesimal deformation assumption, we can refer the independent variables to the initial undeformed configuration of the plate.

Let us introduce an arbitrary portion  $\Omega' \times [-\frac{h}{2}, \frac{h}{2}]$  of plate, where  $\Omega' \subset \subset \Omega$  is a subdomain of  $\Omega$  with regular boundary. Consider the material fiber  $\{x\} \times [-\frac{h}{2}, \frac{h}{2}]$  for  $x \in \partial\Omega'$  and denote by  $t(x, x_3, e_\alpha) \in \mathbb{R}^3$ ,  $|x_3| \leq \frac{h}{2}$ , the *traction vector* acting on a plane containing the direction of the fiber and orthogonal to the direction  $e_\alpha$ . By Cauchy's Lemma [T] we have  $t(x, x_3, e_\alpha) = T(x, x_3)e_\alpha$ , where  $T(x, x_3) \in \mathbb{M}^3$  is the (symmetric) Cauchy stress tensor at the point  $(x, x_3)$ . Denote by  $n$  the unit outer normal vector to  $\partial\Omega'$  such that  $n \cdot e_3 = 0$ . To simplify the notation, it is convenient to consider  $n$  as a two-dimensional vector belonging to the plane  $x_3 = 0$  containing the middle

surface  $\Omega$  of the plate. By the classical Stress Principle for plates [Vi], we postulate that the two complementary parts  $\Omega'$  and  $\Omega \setminus \Omega'$  interact with one another through a field of force vectors  $R = R(x, n) \in \mathbb{R}^3$  and couple vectors  $M = M(x, n) \in \mathbb{R}^3$  assigned per unit length at  $x \in \partial\Omega'$ . Denoting by

$$R(x, e_\alpha) = \int_{-h/2}^{h/2} t(x, x_3, e_\alpha) dx_3 \quad (3.5)$$

the force vector (per unit length) acting on a direction orthogonal to  $e_\alpha$  and passing through  $x \in \partial\Omega'$ , the contact force  $R(x, n)$  can be expressed as

$$R(x, n) = T^\Omega(x)n, \quad x \in \partial\Omega', \quad (3.6)$$

where the *surface force tensor*  $T^\Omega(x) \in \mathbb{M}^{3 \times 2}$  is given by

$$T^\Omega(x) = R(x, e_\alpha) \otimes e_\alpha, \quad \text{in } \Omega. \quad (3.7)$$

Let  $P = I - e_3 \otimes e_3$  be the projection of  $\mathbb{R}^3$  along the direction  $e_3$ .  $T^\Omega$  is decomposed additively by  $P$  in its *membranal* and *shearing* component

$$T^\Omega = PT^\Omega + (I - P)T^\Omega \equiv T^{\Omega(m)} + T^{\Omega(s)}, \quad (3.8)$$

where, following the standard nomenclature in plate theory, the components  $T_{\alpha\beta}^{\Omega(m)} (= T_{\beta\alpha}^{\Omega(m)})$ ,  $\alpha, \beta = 1, 2$ , are called the *membrane forces* and the components  $T_{3\beta}^{\Omega(s)}$ ,  $\beta = 1, 2$ , are the *shear forces* (also denoted as  $T_{3\beta}^{\Omega(s)} = Q_\beta$ ). The assumption of infinitesimal deformations and the hypothesis of vanishing in-plane displacements of the middle surface of the plate allow us to take

$$T^{\Omega(m)} = 0, \quad \text{in } \Omega. \quad (3.9)$$

Denote by

$$M(x, e_\alpha) = \int_{-h/2}^{h/2} x_3 e_3 \times t(x, x_3, e_\alpha) dx_3, \quad \alpha = 1, 2, \quad (3.10)$$

the contact couple acting at  $x \in \partial\Omega'$  on a direction orthogonal to  $e_\alpha$  passing through  $x$ . Note that  $M(x, e_\alpha) \cdot e_3 = 0$  by definition, that is  $M(x, e_\alpha)$  actually is a two-dimensional couple field belonging to the middle plane of the plate. Analogously to (3.6), we have

$$M(x, n) = M^\Omega(x)n, \quad x \in \partial\Omega', \quad (3.11)$$

where the *surface couple tensor*  $M^\Omega(x) \in \mathbb{M}^{2 \times 2}$  has the expression

$$M^\Omega(x) = M(x, e_\alpha) \otimes e_\alpha. \quad (3.12)$$

A direct calculation shows that

$$M(x, e_\alpha) = e_3 \times e_\beta M_{\beta\alpha}(x), \quad (3.13)$$

where

$$M_{\beta\alpha}(x) = \int_{-h/2}^{h/2} x_3 T_{\beta\alpha}(x, x_3) dx_3, \quad \alpha, \beta = 1, 2, \quad (3.14)$$

are the *bending moments* (for  $\alpha = \beta$ ) and the *twisting moments* (for  $\alpha \neq \beta$ ) of the plate at  $x$  (per unit length).

The differential equilibrium equation for the plate follows from the integral mechanical balance equations applied to any subdomain  $\Omega' \subset \subset \Omega$  [T]. Denote by  $q(x)e_3$  the external transversal force per unit area acting in  $\Omega$ . The statical equilibrium of the plate is satisfied if and only if the following two equations are simultaneously satisfied:

$$\begin{cases} \int_{\partial\Omega'} T^\Omega n ds + \int_{\Omega'} q e_3 dx = 0, \\ \int_{\partial\Omega'} ((x - x_0) \times T^\Omega n + M^\Omega n) ds + \int_{\Omega'} (x - x_0) \times q e_3 dx = 0, \end{cases} \quad (3.15)$$

for every subdomain  $\Omega' \subseteq \Omega$ , where  $x_0$  is a fixed point. By applying the Divergence Theorem in  $\Omega'$  and by the arbitrariness of  $\Omega'$  we deduce

$$\begin{cases} \operatorname{div}_x T^{\Omega(s)} + q e_3 = 0, & \text{in } \Omega, \\ \operatorname{div}_x M^\Omega + (T^{\Omega(s)})^T e_3 \times e_3 = 0, & \text{in } \Omega. \end{cases} \quad (3.17)$$

Consider the case in which the boundary of the plate  $\partial\Omega$  is subjected simultaneously to a couple field  $\widehat{M}$ ,  $\widehat{M} \cdot e_3 = 0$ , and a transversal force field  $\widehat{Q}e_3$ . Local equilibrium considerations on points of  $\partial\Omega$  yield the following boundary conditions:

$$\begin{cases} M^\Omega n = \widehat{M}, & \text{on } \partial\Omega, \\ T^{\Omega(s)} n = \widehat{Q}e_3, & \text{on } \partial\Omega. \end{cases} \quad (3.19)$$

where  $n$  is the unit outer normal to  $\partial\Omega$ . In cartesian components, the equilibrium equations (3.17)–(3.20) take the form

$$\begin{cases} M_{\alpha\beta,\beta} - Q_\alpha = 0, & \text{in } \Omega, \alpha = 1, 2, \\ Q_{\alpha,\alpha} + q = 0, & \text{in } \Omega, \\ M_{\alpha\beta} n_\alpha n_\beta = \widehat{M}_n, & \text{on } \partial\Omega, \\ M_{\alpha\beta} \tau_\alpha n_\beta = -\widehat{M}_\tau, & \text{on } \partial\Omega, \\ Q_\alpha n_\alpha = \widehat{Q}, & \text{on } \partial\Omega. \end{cases} \quad (3.21)$$

$$Q_{\alpha,\alpha} + q = 0, \quad \text{in } \Omega, \quad (3.22)$$

$$M_{\alpha\beta} n_\alpha n_\beta = \widehat{M}_n, \quad \text{on } \partial\Omega, \quad (3.23)$$

$$M_{\alpha\beta} \tau_\alpha n_\beta = -\widehat{M}_\tau, \quad \text{on } \partial\Omega, \quad (3.24)$$

$$Q_\alpha n_\alpha = \widehat{Q}, \quad \text{on } \partial\Omega. \quad (3.25)$$

Here, following a standard convention in the theory of plates, we have decomposed the boundary couple field  $\widehat{M}$  in local coordinates as  $\widehat{M} = \widehat{M}_\tau n + \widehat{M}_n \tau$ .

To complete the formulation of the equilibrium problem, we need to introduce the constitutive equation of the material. We limit ourselves to the Kirchhoff-Love theory and we choose to regard the kinematical assumptions  $E_{i3}[u] = 0$ ,  $i = 1, 2, 3$  (see (3.3), with  $\gamma = 0$ ) as internal constraints, that is we restrict the possible deformations of the points of the plate to those whose infinitesimal strain tensor belongs to the set

$$\mathcal{M} = \{E \in \mathbb{M}^{3 \times 3} | E = E^T, E \cdot A = 0, \text{ for } A = e_i \otimes e_3 + e_3 \otimes e_i, i = 1, 2, 3\}. \quad (3.26)$$

Then, by the *Generalized Principle of Determinism* [T], the Cauchy stress tensor  $T$  at any point  $(x, x_3)$  of the plate is additively decomposed in an *active* (symmetric) part  $T_A$  and in a *reactive* (symmetric) part  $T_R$ :

$$T = T_A + T_R, \quad (3.27)$$

where  $T_R$  does not work in any admissible motion, e.g.,  $T_R \in \mathcal{M}^\perp$ . Consistently with the Principle, the active stress  $T_A$  belongs to  $\mathcal{M}$  and, in cartesian coordinates, we have

$$T_A = T_{A\alpha\beta} e_\alpha \otimes e_\beta, \quad \alpha, \beta = 1, 2, \quad (3.28)$$

$$T_R = T_{R\alpha 3} e_\alpha \otimes e_3 + T_{R3\alpha} e_3 \otimes e_\alpha + T_{R33} e_3 \otimes e_3. \quad (3.29)$$

In linear theory, on assuming the reference configuration unstressed, the active stress in a point  $(x, x_3)$  of the plate,  $x \in \overline{\Omega}$  and  $|x_3| \leq h/2$ , is given by a linear mapping from  $\mathcal{M}$  into itself by means of the fourth order *elasticity tensor*  $\mathbb{C}_\mathcal{M}$ :

$$T_A = \mathbb{C}_\mathcal{M} E[u]. \quad (3.30)$$

Here, in view of (3.26) and (3.28),  $\mathbb{C}_\mathcal{M}$  can be assumed to belong to  $\mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$ . Moreover, we assume that  $\mathbb{C}_\mathcal{M}$  is constant over the thickness of the plate and satisfies the minor and major symmetry conditions expressed in cartesian coordinates as (we drop the subscript  $\mathcal{M}$ )

$$C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta} = C_{\alpha\beta\delta\gamma} = C_{\gamma\delta\alpha\beta}, \quad \alpha, \beta, \gamma, \delta = 1, 2, \quad \text{in } \Omega. \quad (3.31)$$

We refer to [PG] and [L-PG] for a representation formula of  $\mathbb{C}$  based on the maximal response symmetry of the material compatible with the internal constraints.

Using (3.27) and recalling (3.9), we obtain the corresponding decomposition for  $T^\Omega$  and  $M^\Omega$ :

$$T^\Omega = T_R^{\Omega(s)}, \quad M^\Omega = M_A^\Omega, \quad (3.32)$$

that is the shear forces and the moments have reactive and active nature, respectively. By (3.30), after integration over the thickness, the surface couple tensor is given by

$$M^\Omega(x) = -\frac{h^3}{12}\mathcal{E}\mathbb{C}(x)(\nabla_x^2 w(x)), \quad \text{in } \Omega, \quad (3.33)$$

where  $\mathcal{E} \in \mathbb{M}^2$  has cartesian components  $\mathcal{E}_{11} = \mathcal{E}_{11} = 0$ ,  $\mathcal{E}_{12} = -1$ ,  $\mathcal{E}_{21} = 1$ . Constitutive equation (3.33) can be written in more expressive way in terms of the bending and twisting moments as follows:

$$M_{\alpha\beta}(w) = -P_{\alpha\beta\gamma\delta}(x)w_{,\gamma\delta}, \quad \alpha, \beta = 1, 2, \quad (3.34)$$

where

$$\mathbb{P}(x) = \frac{h^3}{12}\mathbb{C}(x), \quad \text{in } \Omega, \quad (3.35)$$

is the *plate elasticity tensor*. Combining (3.17) and (3.18), and by eliminating the reactive term  $T_R^{\Omega(s)}$ , we obtain the classical partial differential equation of the Kirchhoff-Love's bending theory of thin elastic plates, that, written in cartesian coordinates, takes the form

$$(P_{\alpha\beta\gamma\delta}(x)w_{,\gamma\delta}(x))_{,\alpha\beta} = q, \quad \text{in } \Omega. \quad (3.36)$$

In the remaining part of this section we complete the formulation of the equilibrium problem for a Kirchhoff-Love plate by writing the boundary conditions corresponding to (3.23)–(3.25). The determination of these boundary conditions is not a trivial issue because, first, a constitutive equation for shear forces is not available since these have a reactive nature (see (3.32)), and, second, because the three mechanical boundary conditions (3.23)–(3.25) should reasonably "collapse" into two independent boundary conditions for the fourth order partial differential equation (3.36). To this aim, under the additional assumption of  $\mathbb{C}$  positive definite, we adopt a variational approach and we impose the stationarity condition on the *total potential energy* functional  $J$  of the plate. Consider the space of regular kinematically admissible displacements

$$\mathcal{D} = \{v : \Omega \times (-h/2, h/2) \rightarrow \mathbb{R}^3 \mid v(x, x_3) = \eta(x)e_3 - x_3 \nabla_x \eta(x), \text{ with } \eta : \Omega \rightarrow \mathbb{R}\}. \quad (3.37)$$

The energy functional  $J : \mathcal{D} \rightarrow \mathbb{R}$  is defined as

$$J(v) = a(v, v) - l(v), \quad (3.38)$$

where  $a(v, v)$  is interpreted as the *elastic energy* stored in the plate for the displacement field  $v$  and  $l(v)$  is the *load potential* that accounts for the energy

of the system of applied loads  $q, \widehat{M}, \widehat{Q}$ . The three-dimensional expression of the elastic energy in the Linear Theory of Elasticity is given by

$$a(v, v) = \frac{1}{2} \int_{-h/2}^{h/2} \int_{\Omega} \mathbb{C} E[v] \cdot E[v] dx dx_3, \quad (3.39)$$

where, in view of (3.37),  $E[v] = -x_3 \nabla_x^2 \eta(x)$ . After integration over the thickness, we obtain

$$a(v, v) = -\frac{1}{2} \int_{\Omega} M_{\alpha\beta}(\eta) \eta_{,\alpha\beta} dx, \quad (3.40)$$

where  $M_{\alpha\beta}(\eta)$  are as in (3.34) with  $w$  replaced by  $\eta$ . The load functional has the expression

$$l(v) = \int_{\Omega} q \eta dx + \int_{\partial\Omega} (\widehat{Q} \eta + \widehat{M}_2 \eta_{,2} - \widehat{M}_1 \eta_{,1}) ds. \quad (3.41)$$

Then, the stationarity condition (in fact, minimum condition) on  $J$  at  $w$  yields

$$\int_{\Omega} M_{\alpha\beta}(w) \eta_{,\alpha\beta} dx + \int_{\Omega} q \eta + \int_{\partial\Omega} (\widehat{Q} \eta + \widehat{M}_2 \eta_{,2} - \widehat{M}_1 \eta_{,1}) ds = 0, \quad (3.42)$$

for every regular function  $\eta$ . Integrating by parts twice on the first integral we obtain

$$\begin{aligned} \int_{\Omega} (M_{\alpha\beta,\alpha\beta}(w) + q) \eta dx + \int_{\partial\Omega} (-M_{\alpha\beta,\beta}(w) n_{\alpha} + \widehat{Q}) \eta ds + \\ + \int_{\partial\Omega} (M_{\alpha\beta}(w) n_{\beta} \eta_{,\alpha} + \widehat{M}_2 \eta_{,2} - \widehat{M}_1 \eta_{,1}) ds = 0. \end{aligned} \quad (3.43)$$

We elaborate the last integral  $I_1$  of (3.43) by rewriting the first order derivatives of  $\eta$  on  $\partial\Omega$  in terms of the normal and arc-length derivative of  $\eta$ . We have

$$\begin{aligned} I_1 = \int_{\partial\Omega} (M_{\alpha\beta}(w) n_{\beta} n_{\alpha} - \widehat{M}_1 n_1 + \widehat{M}_2 n_2) \eta_{,n} ds + \\ + \int_{\partial\Omega} (M_{\alpha\beta}(w) n_{\beta} \tau_{\alpha} + \widehat{M}_1 n_2 + \widehat{M}_2 n_1) \eta_{,s} ds = I'_1 + I''_1 \end{aligned} \quad (3.44)$$

and, integrating by parts on  $\partial\Omega$ , we get

$$\begin{aligned} I'_1 = (M_{\alpha\beta}(w) n_{\beta} \tau_{\alpha} + \widehat{M}_1 n_2 + \widehat{M}_2 n_1) \eta \Big|_{s=0}^{s=l(\partial\Omega)} - \\ - \int_{\partial\Omega} (M_{\alpha\beta}(w) n_{\beta} \tau_{\alpha} + \widehat{M}_1 n_2 + \widehat{M}_2 n_1)_{,s} \eta ds, \end{aligned} \quad (3.45)$$



where  $l(\partial\Omega)$  is the length of  $\partial\Omega$ . Since  $\partial\Omega$  is of class  $C^{1,1}$ , the boundary term on the right end side of (3.45) identically vanishes. Therefore, the stationarity condition of  $J$  at  $w$  takes the final form

$$\begin{aligned} & \int_{\Omega} (M_{\alpha\beta,\alpha\beta}(w) + q) \eta dx + \\ & \int_{\partial\Omega} \left( -(M_{\alpha\beta}(w)n_{\beta}\tau_{\alpha})_{,s} - M_{\alpha\beta,\beta}(w)n_{\alpha} + \widehat{Q} - (\widehat{M}_1 n_2 + \widehat{M}_2 n_1)_{,s} \right) \eta ds + \\ & + \int_{\partial\Omega} \left( M_{\alpha\beta}(w)n_{\beta}n_{\alpha} - \widehat{M}_1 n_1 + \widehat{M}_2 n_2 \right) \eta_{,n} ds = 0 \end{aligned} \quad (3.46)$$

for every  $\eta \in C^{\infty}(\overline{\Omega}, \mathbb{R})$ . By the arbitrariness of the function  $\eta$ , and of the traces of  $\eta$  and  $\eta_{,n}$  on  $\partial\Omega$ , we determine the equilibrium equation (3.36) and the desired Neumann boundary conditions on  $\partial\Omega$ :

$$M_{\alpha\beta}(w)n_{\alpha}n_{\beta} = \widehat{M}_n, \quad (3.47)$$

$$M_{\alpha\beta,\beta}(w)n_{\alpha} + (M_{\alpha\beta}(w)n_{\beta}\tau_{\alpha})_{,s} = \widehat{Q} - (\widehat{M}_{\tau})_{,s}. \quad (3.48)$$

## 4 Uniqueness and stability of extreme inclusions and free boundaries

### 4.1 Rigid inclusions: uniqueness

In the sequel we shall assume that the plate is made of nonhomogeneous linear elastic material with plate tensor

$$\mathbb{P} = \frac{h^3}{12} \mathbb{C}, \quad (4.1)$$

where the elasticity tensor  $\mathbb{C}(x) \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$  has cartesian components  $C_{ijkl}$  which satisfy the following symmetry conditions

$$C_{ijkl} = C_{klij} = C_{klji} \quad i, j, k, l = 1, 2, \text{ a.e. in } \Omega, \quad (4.2)$$

and, for simplicity, is defined in all of  $\mathbb{R}^2$ .

We make the following assumptions:

I) *Regularity*

$$\mathbb{C} \in C^{1,1}(\mathbb{R}^2, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)), \quad (4.3)$$

II) *Ellipticity (strong convexity)* There exists  $\gamma > 0$  such that

$$\mathbb{C}A \cdot A \geq \gamma|A|^2, \quad \text{in } \mathbb{R}^2, \quad (4.4)$$

for every  $2 \times 2$  symmetric matrix  $A$ .

III) *Dichotomy condition*

$$\text{either} \quad \mathcal{D}(x) > 0, \quad \text{for every } x \in \mathbb{R}^2, \quad (4.5a)$$

$$\text{or} \quad \mathcal{D}(x) = 0, \quad \text{for every } x \in \mathbb{R}^2, \quad (4.5b)$$

where

$$\mathcal{D}(x) = \frac{1}{a_0} |\det S(x)|, \quad (4.6)$$

$$S(x) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \end{pmatrix}, \quad (4.7)$$

$$a_0 = A_0, \quad a_1 = 4C_0, \quad a_2 = 2B_0 + 4E_0, \quad a_3 = 4D_0, \quad a_4 = F_0. \quad (4.8)$$

and

$$\begin{cases} C_{1111} = A_0, & C_{1122} = C_{2211} = B_0, \\ C_{1112} = C_{1121} = C_{1211} = C_{2111} = C_0, \\ C_{2212} = C_{2221} = C_{1222} = C_{2122} = D_0, \\ C_{1212} = C_{1221} = C_{2112} = C_{2121} = E_0, \\ C_{2222} = F_0, \end{cases} \quad (4.9)$$

*Remark 4.1.* Whenever (4.5a) holds we denote

$$\delta_1 = \min_{\mathbb{R}^2} \mathcal{D}. \quad (4.10)$$

We emphasize that, in all the following statements, whenever a constant is said to depend on  $\delta_1$  (among other quantities) it is understood that such dependence occurs *only* when (4.5a) holds.

On the assigned couple field  $\widehat{M}$  let us require the following assumptions:

$$\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2), \quad (\widehat{M}_n, \widehat{M}_{\tau,s}) \neq 0, \quad (4.11)$$

$$\int_{\partial\Omega} \widehat{M}_i = 0, \quad i = 1, 2. \quad (4.12)$$

**Theorem 4.2** (Unique determination of a rigid inclusion with one measurement). *Let  $\Omega$  be a simply connected domain in  $\mathbb{R}^2$  such that  $\partial\Omega$  is of class  $C^{1,1}$  and let  $D_i$ ,  $i = 1, 2$ , be two simply connected domains compactly contained in  $\Omega$ , such that  $\partial D_i$  is of class  $C^{3,1}$ ,  $i = 1, 2$ . Moreover, let  $\Gamma$  be a nonempty open portion of  $\partial\Omega$ , of class  $C^{3,1}$ . Let the plate tensor  $\mathbb{P}$  be given by (4.1), and satisfying (4.2)–(4.4) and the dichotomy condition (4.5a) or (4.5b). Let  $\widehat{M}$  be a boundary couple field satisfying (4.11)–(4.12). Let  $w_i$ ,  $i = 1, 2$ , be the solutions to the mixed problem (1.9)–(1.13), coupled with (1.6), with  $D = D_i$ .*

*If there exists  $g \in \mathcal{A}$  such that*

$$w_1 - w_2 = g, \quad (w_1 - w_2)_{,n} = g_{,n}, \quad \text{on } \Gamma, \quad (4.13)$$

*then*

$$D_1 = D_2. \quad (4.14)$$

*Proof of Theorem 4.2.* Let  $G$  be the connected component of  $\Omega \setminus (\overline{D_1 \cup D_2})$  such that  $\Gamma \subset \partial G$ . Let us notice that, since  $w_i$  satisfies homogeneous Dirichlet conditions on the  $C^{3,1}$  boundary  $\partial D_i$ , by regularity results we have that  $w_i \in H^4(\widetilde{\Omega} \setminus D_i)$ , for every  $\widetilde{\Omega}$ ,  $D_i \subset \subset \widetilde{\Omega} \subset \subset \Omega$ ,  $i = 1, 2$  (see, for example, [Ag]). By Sobolev embedding theorems (see, for instance, [Ad]), we have that  $w_i$  and  $\nabla w_i$  are continuous up to  $\partial D_i$ ,  $i = 1, 2$ . Therefore

$$w_i \equiv 0, \quad \nabla w_i^e \equiv 0, \quad \text{on } \partial D_i. \quad (4.15)$$

Let  $w = w_1 - w_2 - g$ , with  $g(x_1, x_2) = ax_1 + bx_2 + c$ . By our assumptions,  $w$  takes homogeneous Cauchy data on  $\Gamma$ . From the uniqueness of the solution to the Cauchy problem (see, for instance, Theorem 3.8 in [M-R-Ve4]) and also Remark 4 in [M-R-Ve2]) and from the weak unique continuation property, we have that

$$w \equiv 0, \quad \text{in } G.$$

Let us prove, for instance, that  $D_2 \subset D_1$ . We have

$$D_2 \setminus \overline{D_1} \subset \Omega \setminus (\overline{D_1 \cup G}),$$

$$\partial(\Omega \setminus (\overline{D_1 \cup G})) = \Sigma_1 \cup \Sigma_2,$$

where  $\Sigma_2 = \partial D_2 \cap \partial G$ ,  $\Sigma_1 = \partial(\Omega \setminus (\overline{D_1 \cup G})) \setminus \Sigma_2 \subset \partial D_1$ .

Let us distinguish two cases

- i)  $\partial D_1 \cap \Sigma_2 \neq \emptyset$ ,
- ii)  $\partial D_1 \cap \Sigma_2 = \emptyset$ .

In case i), there exists  $P_0 \in \partial D_1 \cap \Sigma_2$ . Since  $w_i(P_0) = 0$ ,  $w(P_0) = 0$ , we have  $g(P_0) = 0$ .

Let  $P_n \in G$ ,  $P_n \rightarrow P_0$ . We have

$$\nabla w(P_n) = 0,$$

$$0 = \lim_{n \rightarrow \infty} \nabla w(P_n) = \nabla w_1^e(P_0) - \nabla w_2^e(P_0) - (a, b) = -(a, b),$$

so that  $g \equiv c$ , and, since  $g(P_0) = 0$ ,

$$g \equiv 0 \quad \Rightarrow \quad w_1 \equiv w_2, \text{ in } G, \quad \Rightarrow \quad \begin{cases} w_1 \equiv w_2 = 0, & \text{on } \Sigma_2, \\ \nabla w_1 \equiv \nabla w_2 = 0, & \text{on } \Sigma_2. \end{cases} \quad (4.16)$$

Integrating by parts equation  $(\operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w_1)))w_1 = 0$ , we obtain

$$\begin{aligned} \int_{\Omega \setminus (\overline{D_1 \cup G})} \mathbb{P}\nabla^2 w_1 \cdot \nabla^2 w_1 &= \int_{\Sigma_1 \cup \Sigma_2} (\mathbb{P}\nabla^2 w_1 \nu \cdot \nu) w_{1,n} + \\ &+ \int_{\Sigma_1 \cup \Sigma_2} (\mathbb{P}\nabla^2 w_1 \nu \cdot \tau) w_{1,s} - \int_{\Sigma_1 \cup \Sigma_2} (\operatorname{div}(\mathbb{P}\nabla^2 w_1) \cdot \nu) w_1, \end{aligned} \quad (4.17)$$

where  $\nu$  is the outer unit normal to  $\Omega \setminus (\overline{D_1 \cup G})$ . By

$$w_1 = 0, \quad \nabla w_1 = 0, \quad \text{on } \Sigma_1,$$

and by (4.16), we have

$$w_1 = w_2 = 0, \quad \nabla w_1 = \nabla w_2 = 0, \quad \text{on } \Sigma_2,$$

so that

$$0 = \int_{\Omega \setminus (\overline{D_1 \cup G})} \mathbb{P}\nabla^2 w_1 \cdot \nabla^2 w_1 \geq \gamma \int_{D_2 \setminus \overline{D_1}} |\nabla^2 w_1|^2. \quad (4.18)$$

If  $D_2 \setminus \overline{D_1} \neq \emptyset$ , then, by the weak unique continuation principle,  $w_1$  coincides with an affine function in  $\Omega \setminus \overline{D_1}$ , contradicting the choice of the nontrivial Neumann data  $\widehat{M}$  on  $\partial\Omega$ . Therefore  $D_2 \subset \overline{D_1}$  and, by the regularity of  $D_i$ ,  $i = 1, 2$ ,  $D_2 \subset D_1$ .

In case ii), either  $\overline{D_1} \cap \overline{D_2} = \emptyset$  or  $\overline{D_1} \subset D_2$ .

Let us consider for instance the first case, the proof of the second case being similar. Integrating by parts, we have

$$\begin{aligned} \int_{D_2} \mathbb{P}\nabla^2 w_1 \cdot \nabla^2 w_1 &= \int_{D_2} \mathbb{P}\nabla^2 w_1 \cdot \nabla^2 (w_1 - g) = \\ &= \int_{\partial D_2} (\mathbb{P}\nabla^2 w_1 \nu \cdot \nu) (w_1 - g)_n + \\ &+ \int_{\partial D_2} (\mathbb{P}\nabla^2 w_1 \nu \cdot \tau) (w_1 - g)_s - \int_{\partial D_2} (\operatorname{div}(\mathbb{P}\nabla^2 w_1) \cdot \nu) (w_1 - g). \end{aligned}$$

By the regularity of  $\partial D_2$ , we may rewrite it as

$$\begin{aligned} \int_{D_2} \mathbb{P} \nabla^2 w_1 \cdot \nabla^2 w_1 &= \int_{\partial D_2} (\mathbb{P} \nabla^2 w_1 \nu \cdot \nu) (w_1 - g)_{,n} + \\ &\quad - \int_{\partial D_2} ((\mathbb{P} \nabla^2 w_1 \nu \cdot \tau)_{,s} + \operatorname{div}(\mathbb{P} \nabla^2 w_1) \cdot \nu) (w_1 - g), \end{aligned}$$

Recalling that  $w_1 - g = w_2 = 0$ ,  $(w_1 - g)_{,n} = w_{2,n} = 0$  on  $\partial D_2$ , we have that

$$\int_{D_2} \mathbb{P} \nabla^2 w_1 \cdot \nabla^2 w_1 = 0.$$

If  $D_2 \neq \emptyset$ , then  $w_1$  coincides with an affine function in  $D_2$ , and, by the weak unique continuation principle, also in  $\Omega \setminus \overline{D_1}$ , contradicting the choice of a nontrivial  $\widehat{M}$ . Therefore  $D_2 = \emptyset$ . Symmetrically, we obtain that  $D_1 = \emptyset$ , that is  $D_1 = D_2$ .  $\square$

## 4.2 Rigid inclusions: stability

In order to prove the stability estimates, we need the following further quantitative assumptions.

Given  $\rho_0, M_0, M_1 > 0$ , we assume that

$$|\Omega| \leq M_1 \rho_0^2, \quad (4.19)$$

$$\operatorname{dist}(D, \partial\Omega) \geq \rho_0, \quad (4.20)$$

$$\partial\Omega \text{ is of class } C^{2,1} \text{ with constants } \rho_0, M_0, \quad (4.21)$$

$$\Gamma \text{ is of class } C^{3,1} \text{ with constants } \rho_0, M_0, \quad (4.22)$$

$$\partial D \text{ is of class } C^{3,1} \text{ with constants } \rho_0, M_0, \quad (4.23)$$

where  $|\Omega|$  denotes the Lebesgue measure of  $\Omega$ . Moreover, we assume that for some  $P_0 \in \Sigma$  and some  $\delta_0, 0 < \delta_0 < 1$ ,

$$\partial\Omega \cap R_{\frac{\rho_0}{M_0}, \rho_0}(P_0) \subset \Gamma, \quad (4.24)$$

and that

$$|\Gamma| \leq (1 - \delta_0) |\partial\Omega|. \quad (4.25)$$

On the Neumann data  $\widehat{M}$  we assume that

$$\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2), \quad (\widehat{M}_n, (\widehat{M}_\tau)_{,s}) \neq 0, \quad (4.26)$$

$$\int_{\partial\Omega} \widehat{M}_i = 0, \quad i = 1, 2, \quad (4.27)$$

$$\text{supp}(\widehat{M}) \subset\subset \Gamma, \quad (4.28)$$

and that, for a given constant  $F > 0$ ,

$$\frac{\|\widehat{M}\|_{L^2(\partial\Omega, \mathbb{R}^2)}}{\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}} \leq F. \quad (4.29)$$

On the elasticity tensor  $\mathbb{C}$ , we assume the same a priori information made in Subsection 4.1, and we introduce a parameter  $M > 0$  such that

$$\sum_{i,j,k,l=1}^2 \sum_{m=0}^2 \rho_0^m \|\nabla^m C_{ijkl}\|_{L^\infty(\mathbb{R}^2)} \leq M. \quad (4.30)$$

We shall refer to the set of constants  $M_0, M_1, \delta_0, F, \gamma, M, \delta_1$  as the *a priori data*. The scale parameter  $\rho_0$  will appear explicitly in all formulas, whereas the dependence on the thickness parameter  $h$  will be omitted.

**Theorem 4.3** (Stability result). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  satisfying (4.19) and (4.21). Let  $D_i, i = 1, 2$ , be two simply connected open subsets of  $\Omega$  satisfying (4.20) and (4.23). Moreover, let  $\Gamma$  be an open portion of  $\partial\Omega$  satisfying (4.22), (4.24) and (4.25). Let  $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$  satisfy (4.26)–(4.29) and let the plate tensor  $\mathbb{P}$  given by (4.1) satisfy (4.2), (4.30), (4.4) and the dichotomy condition. Let  $w_i \in H^2(\Omega \setminus \overline{D_i})$  be the solution to (1.9)–(1.13), coupled with (1.6), when  $D = D_i, i = 1, 2$ . If, given  $\epsilon > 0$ , we have*

$$\min_{g \in \mathcal{A}} \left\{ \|w_1 - w_2 - g\|_{L^2(\Sigma)} + \rho_0 \|(w_1 - w_2 - g)_{,n}\|_{L^2(\Sigma)} \right\} \leq \epsilon, \quad (4.31)$$

then we have

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq C \rho_0 (\log |\log \tilde{\epsilon}|)^{-\eta}, \quad 0 < \tilde{\epsilon} < e^{-1}, \quad (4.32)$$

and

$$d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}) \leq C \rho_0 (\log |\log \tilde{\epsilon}|)^{-\eta}, \quad 0 < \tilde{\epsilon} < e^{-1}, \quad (4.33)$$

where  $\tilde{\epsilon} = \frac{\epsilon}{\rho_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}}}$  and  $C, \eta, C > 0, 0 < \eta \leq 1$ , are constants only depending on the a priori data.

*Proof of Theorem 4.3.* Let us rough-out a sketch of the proof, referring the interested reader to Section 3 of [M-R-Ve5]. Retracing the proof of the uniqueness theorem, the basic idea is that of deriving the quantitative version in the stability context of the vanishing of the integral  $\int_{\Omega \setminus (\overline{D_i \cup G})} \mathbb{P} \nabla^2 w_i \cdot \nabla^2 w_i$ , for  $i = 1, 2$ , that is a control with some small term emerging from the bound (4.31) on the Cauchy data. To this aim, we need stability estimates of continuation from Cauchy data and propagation of smallness estimates. In particular, the propagation of smallness of  $|\nabla^2 w_i|$ , for  $i = 1, 2$ , from a neighborhood of  $\Gamma$  towards  $\partial(\Omega \setminus (\overline{D_i \cup G}))$  is performed through iterated application of the three sphere inequality (1.18) over suitable chains of disks. A strong hindrance which occurs in this step is related to the difficulty of getting arbitrarily closer to the boundary of the set  $\Omega \setminus (\overline{D_i \cup G})$ , due to the absence, in our general setting, of any a priori information on the reciprocal position of  $D_1$  and  $D_2$ . For this reason, as a preparatory step, we derive the following rough estimate

$$\max \left\{ \int_{D_2 \setminus \overline{D_1}} |\nabla^2 w_1|^2, \int_{D_1 \setminus \overline{D_2}} |\nabla^2 w_2|^2 \right\} \leq \rho_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}}^2 (\log |\log \tilde{\epsilon}|)^{-\frac{1}{2}}, \quad (4.34)$$

which holds for every  $\tilde{\epsilon} < e^{-1}$ , with  $\tilde{\epsilon} = \frac{\epsilon}{\rho_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}}}$ , and where  $C > 0$  depends only on  $\gamma$ ,  $M$ ,  $\delta_1$ ,  $M_0$ ,  $M_1$  and  $\delta_0$ .

Since a pointwise lower bound for  $|\nabla^2 w_i|^2$  cannot hold in general, the next crucial step consists in the following Claim.

**Claim.** If

$$\max \left\{ \int_{D_2 \setminus \overline{D_1}} |\nabla^2 w_1|^2, \int_{D_1 \setminus \overline{D_2}} |\nabla^2 w_2|^2 \right\} \leq \frac{\eta}{\rho_0^2}, \quad (4.35)$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq C \rho_0 \left[ \log \left( \frac{C \rho_0^4 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^n)}^2}{\eta} \right) \right]^{-\frac{1}{B}}, \quad (4.36)$$

where  $B > 0$  and  $C > 0$  only depend on  $\gamma$ ,  $M$ ,  $\delta_1$ ,  $M_0$ ,  $M_1$ ,  $\delta_0$  and  $F$ .

*Proof of the Claim.* Denoting for simplicity  $d = d_{\mathcal{H}}(\partial D_1, \partial D_2)$ , we may assume, with no loss of generality, that there exists  $x_0 \in \partial D_1$  such that  $\text{dist}(x_0, \partial D_2) = d$ . Let us distinguish two cases:

i)  $B_d(x_0) \subset D_2$ ;

ii)  $B_d(x_0) \cap D_2 = \emptyset$ .

In case i), by the regularity assumptions made on  $\partial D_1$ , there exists  $x_1 \in D_2 \setminus D_1$  such that  $B_{td}(x_1) \subset D_2 \setminus D_1$ , with  $t = \frac{1}{1+\sqrt{1+M_0^2}}$ .

In [M-R-Ve6], by iterated application of the three sphere inequality (1.18), we have obtained the following *Lipschitz propagation of smallness* estimate: there exists  $s > 1$ , only depending on  $\gamma$ ,  $M$ ,  $\delta_1$ ,  $M_0$  and  $\delta_0$ , such that for every  $\rho > 0$  and every  $\bar{x} \in (\Omega \setminus \overline{D})_{s\rho}$ , we have

$$\int_{B_\rho(\bar{x})} |\nabla^2 w|^2 \geq \frac{C\rho_0^2}{\exp \left[ A \left( \frac{\rho_0}{\rho} \right)^B \right]} \|\widehat{M}\|_{H^{-\frac{1}{2}}}^2, \quad (4.37)$$

where  $A > 0$ ,  $B > 0$  and  $C > 0$  only depend on  $\gamma$ ,  $M$ ,  $\delta_1$ ,  $M_0$ ,  $M_1$ ,  $\delta_0$  and  $F$ . By (4.35) and by applying (4.37) with  $\rho = \frac{td}{s}$ , we have

$$\eta \geq \frac{C\rho_0^4}{\exp \left[ A \left( \frac{s\rho_0}{td} \right)^B \right]} \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}^2, \quad (4.38)$$

where  $A > 0$ ,  $B > 0$  and  $C > 0$  only depend on  $\gamma$ ,  $M$ ,  $\delta_1$ ,  $M_0$ ,  $M_1$ ,  $\delta_0$  and  $F$ .

By (4.38) we easily find (4.36).

Case ii) can be treated similarly by substituting  $w_1$  with  $w_2$ .  $\square$

By applying the Claim to (4.34), we obtain a first stability estimates of log-log-log type. At this stage, a tool which turns out to be very useful is a geometrical result, firstly stated in [Al-B-R-Ve1], which ensures that there exists  $\epsilon_0 > 0$ , only depending on  $\gamma$ ,  $M$ ,  $\delta_1$ ,  $M_0$ ,  $M_1$ ,  $\delta_0$  and  $F$ , such that if  $\epsilon \leq \epsilon_0$  then  $\partial G$  is of Lipschitz class with constants  $\widetilde{\rho}_0$ ,  $L$ , with  $L$  and  $\frac{\widetilde{\rho}_0}{\rho_0}$  only depending on  $M_0$ . Lipschitz regularity prevents the occurrence of uncontrollable narrowings or cuspidal points in  $G$  and allows to refine the geometrical constructions of the chains of disks to which we apply the three sphere inequality, obtaining the better estimate

$$\max \left\{ \int_{D_2 \setminus \overline{D_1}} |\nabla^2 w_1|^2, \int_{D_1 \setminus \overline{D_2}} |\nabla^2 w_2|^2 \right\} \leq \rho_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}}^2 |\log \widetilde{\epsilon}|^{-\sigma}, \quad (4.39)$$

which holds for every  $\widetilde{\epsilon} < 1$ , where  $C > 0$  and  $\sigma > 0$  depend only on  $\gamma$ ,  $M$ ,  $\delta_1$ ,  $M_0$ ,  $M_1$ ,  $\delta_0$ ,  $L$  and  $\frac{\widetilde{\rho}_0}{\rho_0}$ . Again, by applying the Claim, the desired estimates follow.  $\square$

### 4.3 Cavities and unknown boundary portions: uniqueness

In this subsection we consider the inverse problems of determining unknown boundaries of the following two kinds: i) the boundary of a cavity, ii) an



unknown boundary portion of  $\partial\Omega$ . In both cases, we have homogeneous Neumann conditions on the unknown boundary. Neumann boundary conditions lead to further complications in the arguments involving integration by parts. To give an idea of the differences, given any connected component  $F$  of  $D_2 \setminus \overline{D_1}$ , whose boundary is made of two arcs  $\tau \subset \partial D_1$  and  $\gamma \subset \partial D_2$ , having common endpoints  $P_1$  and  $P_2$ , the analogue of (4.17) becomes

$$\begin{aligned} \int_F \mathbb{P} \nabla^2 w_1 \cdot \nabla^2 w_1 &= \int_\tau (\mathbb{P} \nabla^2 w_1 n^1 \cdot \tau^1 w_1)_{,s} - \int_\gamma (\mathbb{P} \nabla^2 w_2 n^2 \cdot \tau^2 w_2)_{,s} = \\ &= [(\mathbb{P} \nabla^2 w_1 n^1 \cdot \tau^1)(P_1) - (\mathbb{P} \nabla^2 w_1 n^2 \cdot \tau^2)(P_1)] (w_1(P_1) - w_1(P_2)), \end{aligned} \quad (4.40)$$

where  $n^i, \tau^i$  denotes the unit normal and tangent vector to  $D_i$ ,  $i = 1, 2$ .

Since, in general, the boundaries of  $D_1$  and  $D_2$  intersect nontangentially, the above expression does not vanish and the contradiction arguments fails. For this reason, we need two boundary measurements to prove uniqueness, as stated in Theorem 4.4. Instead, uniqueness with one measurement can be restored in the problem of the determination of an unknown boundary portion, by taking advantage of the fact that the two plates have a common regular boundary portion, say  $\Gamma$ , see Theorem 4.5.

Let  $D \subset\subset \Omega$  be a domain of class  $C^{1,1}$  representing an unknown cavity inside the plate  $\Omega$ . Under the same assumptions made in Subsection 4.1, the transversal displacement  $w$  satisfies the following Neumann problem

$$\begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P} \nabla^2 w)) = 0, & \text{in } \Omega \setminus \overline{D}, & (4.41) \\ (\mathbb{P} \nabla^2 w) n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, & (4.42) \\ \operatorname{div}(\mathbb{P} \nabla^2 w) \cdot n + ((\mathbb{P} \nabla^2 w) n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega, & (4.43) \\ (\mathbb{P} \nabla^2 w) n \cdot n = 0, & \text{on } \partial D, & (4.44) \\ \operatorname{div}(\mathbb{P} \nabla^2 w) \cdot n + ((\mathbb{P} \nabla^2 w) n \cdot \tau)_{,s} = 0, & \text{on } \partial D, & (4.45) \end{cases}$$

which admits a solution  $w \in H^2(\Omega \setminus \overline{D})$ , which is uniquely determined up to addition of an affine function.

Concerning the inverse problem of the determination of the cavity  $D$  inside the plate, let us recall the following result.

**Theorem 4.4** (Uniqueness with two boundary measurements). *Let  $\Omega$  be a simply connected domain in  $\mathbb{R}^2$  such that  $\partial\Omega$  is of class  $C^{1,1}$  and let  $D_i$ ,  $i = 1, 2$ , be two simply connected domains compactly contained in  $\Omega$ , such that  $\partial D_i$  is of class  $C^{4,1}$ ,  $i = 1, 2$ . Moreover, let  $\Gamma$  be a nonempty open portion of  $\partial\Omega$ , of class  $C^{3,1}$ . Let the plate tensor  $\mathbb{P}$  be given by (4.1), and satisfying (4.3) (4.4) and the dichotomy condition (4.5a) or (4.5b). Let  $\widehat{M}, \widehat{M}^*$  be two*

boundary couple fields both satisfying (4.11)–(4.12) and such that  $(\widehat{M}_n, \widehat{M}_{\tau,s})$  and  $(\widehat{M}_n^*, \widehat{M}_{\tau,s}^*)$  are linearly independent in  $H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2) \times H^{-\frac{3}{2}}(\partial\Omega, \mathbb{R}^2)$ . Let  $w_i, w_i^*, i = 1, 2$ , be solutions to the Neumann problem (4.41)–(4.45), with  $D = D_i$ , corresponding to boundary data  $\widehat{M}, \widehat{M}^*$  respectively. If

$$w_1 = w_2, \quad w_{1,n} = w_{2,n}, \quad \text{on } \Gamma, \quad (4.46)$$

$$w_1^* = w_2^*, \quad w_{1,n}^* = w_{2,n}^*, \quad \text{on } \Gamma, \quad (4.47)$$

then

$$D_1 = D_2. \quad (4.48)$$

Next, let us consider the case of a plate whose boundary is composed by an accessible portion  $\Gamma$  and by an unknown inaccessible portion  $I$ , to be determined. More precisely, let  $\Gamma, I$  be two closed, nonempty sub-arcs of the boundary  $\partial\Omega$  such that

$$\Gamma \cup I = \partial\Omega, \quad \Gamma \cap I = \{Q, R\}, \quad (4.49)$$

where  $Q, R$  are two distinct points of  $\partial\Omega$ . On the assigned couple field  $\widehat{M}$  let us require the following assumptions:

$$\widehat{M} \in L^2(\Gamma, \mathbb{R}^2), \quad (\widehat{M}_n, \widehat{M}_{\tau,s}) \neq 0, \quad (4.50)$$

$$\int_{\Gamma} \widehat{M}_i = 0, \quad i = 1, 2. \quad (4.51)$$

Under the same assumptions made in Subsection 4.1 for the plate tensor and the domain  $\Omega$ , the transversal displacement  $w \in H^2(\Omega)$  satisfies the following Neumann problem

$$\begin{cases} M_{\alpha\beta,\alpha\beta} = 0, & \text{in } \Omega, \end{cases} \quad (4.52)$$

$$\begin{cases} M_{\alpha\beta} n_{\alpha} n_{\beta} = \widehat{M}_n, & \text{on } \Gamma, \end{cases} \quad (4.53)$$

$$\begin{cases} M_{\alpha\beta,\beta} n_{\alpha} + (M_{\alpha\beta} n_{\beta} \tau_{\alpha})_{,s} = -(\widehat{M}_{\tau})_{,s}, & \text{on } \Gamma, \end{cases} \quad (4.54)$$

$$\begin{cases} M_{\alpha\beta} n_{\alpha} n_{\beta} = 0, & \text{on } I, \end{cases} \quad (4.55)$$

$$\begin{cases} M_{\alpha\beta,\beta} n_{\alpha} + (M_{\alpha\beta} n_{\beta} \tau_{\alpha})_{,s} = 0, & \text{on } I. \end{cases} \quad (4.56)$$

Concerning the inverse problem of the determination of the unknown boundary portion  $I$ , in [M-R3] we have proved the following result.

**Theorem 4.5** (Unique determination of unknown boundaries with one measurement). *Let  $\Omega_1, \Omega_2$  be two simply connected bounded domains in  $\mathbb{R}^2$  such that  $\partial\Omega_i, i = 1, 2$ , are of class  $C^{4,1}$ . Let  $\partial\Omega_i = I_i \cup \Gamma, i = 1, 2$ , where  $I_i$*

and  $\Gamma$  are the inaccessible and the accessible parts of the boundaries  $\partial\Omega_i$ , respectively. Let us assume that  $\Omega_1$  and  $\Omega_2$  lie on the same side of  $\Gamma$  and that conditions (4.49) are satisfied by both pairs  $\{I_1, \Gamma\}$  and  $\{I_2, \Gamma\}$ . Let the plate tensor  $\mathbb{P}$  of class  $C^{2,1}(\mathbb{R}^2)$  be given by (4.1), and satisfying (4.3) (4.4) and the dichotomy condition (4.5a) or (4.5b). Let  $\widehat{M} \in L^2(\Gamma, \mathbb{R}^2)$  be a boundary couple field satisfying conditions (4.50), (4.12). Let  $w_i \in H^2(\Omega_i)$  be a solution to the Neumann problem (4.52)–(4.51) in  $\Omega = \Omega_i$ ,  $i = 1, 2$ . If

$$w_1 = w_2, \quad w_{1,n} = w_{2,n}, \quad \text{on } \Gamma, \quad (4.57)$$

then

$$\Omega_1 = \Omega_2. \quad (4.58)$$

## 5 Size estimates for extreme inclusions

### 5.1 Formulation of the problem and main results

Let us assume that the middle plane of the plate  $\Omega$  is a bounded domain in  $\mathbb{R}^2$  of class  $C^{1,1}$  with constants  $\rho_0$ ,  $M_0$ . In the present section we shall derive constructive upper and lower bounds of the area of either a rigid inclusion or a cavity in an elastic plate from a single boundary measurement. These *extreme* inclusions will be represented by an open subset  $D$  of  $\Omega$  such that  $\Omega \setminus \overline{D}$  is connected and  $D$  is compactly contained in  $\Omega$ ; that is, there is a number  $d_0 > 0$  such that

$$\text{dist}(D, \partial\Omega) \geq d_0\rho_0. \quad (5.1)$$

In addition, in proving the lower bound for the area of  $D$ , we shall introduce the following *a priori* information, which is a way of requiring that  $D$  is not "too thin".

**Definition 5.1.** (Scale Invariant Fatness Condition) Given a domain  $D$  having Lipschitz boundary with constants  $r\rho_0$  and  $L$ , where  $r > 0$ , we shall say that it satisfies the Scale Invariant Fatness Condition with constant  $Q > 0$  if

$$\text{diam}(D) \leq Qr\rho_0. \quad (5.2)$$

*Remark 5.2.* It is evident that if  $D$  satisfies Definition 5.1, then we have the trivial upper and lower estimates

$$\frac{\omega_2}{(1 + \sqrt{1 + L^2})^2} r^2 \rho_0^2 \leq \text{area}(D) \leq \omega_2 Q^2 r^2 \rho_0^2, \quad (5.3)$$

where  $\omega_2$  denotes the measure of the unit disk in  $\mathbb{R}^2$ . Since we are interested in obtaining upper and lower bounds of the area of  $D$  when  $D$  is unknown, it will be necessary to consider also the number  $r$  as an unknown parameter, and all our estimates will not depend on  $r$ . Conversely, the parameters  $L$  and  $Q$ , which are invariant under scaling, will be considered as *a priori* information on the unknown inclusion  $D$ .

For reader's convenience and in order to introduce some useful notation, we briefly recall the formulation of the equilibrium problem when the plate contains either a rigid inclusion or a cavity, and when the inclusion is absent.

Let us assume that the plate tensor  $\mathbb{P} \in L^\infty(\mathbb{R}^2, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$  given by (4.1) satisfies the symmetry conditions (4.2) and the strong convexity condition (4.4). Moreover, let  $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$  satisfy (4.27).

If an inclusion  $D$  made by rigid material is present, with boundary  $\partial D$  of class  $C^{1,1}$ , then the transversal displacement  $w_R$  corresponding to the assigned couple field  $\widehat{M}$  is given as the weak solution  $w_R \in H^2(\Omega \setminus \overline{D})$  of the boundary value problem

$$\begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w_R)) = 0, & \text{in } \Omega \setminus \overline{D}, \\ (\mathbb{P}\nabla^2 w_R)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \operatorname{div}(\mathbb{P}\nabla^2 w_R) \cdot n + ((\mathbb{P}\nabla^2 w_R)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega, \\ w_R|_{\overline{D}} \in \mathcal{A}, & \text{in } \overline{D}, \\ \frac{\partial w_R^e}{\partial n} = \frac{\partial w_R^i}{\partial n}, & \text{on } \partial D, \end{cases} \quad \begin{matrix} (5.4) \\ (5.5) \\ (5.6) \\ (5.7) \\ (5.8) \end{matrix}$$

coupled with the equilibrium conditions for the rigid inclusion  $D$

$$\int_{\partial D} (\operatorname{div}(\mathbb{P}\nabla^2 w_R^e) \cdot n + ((\mathbb{P}\nabla^2 w_R^e)n \cdot \tau)_{,s}) g - ((\mathbb{P}\nabla^2 w_R^e)n \cdot n) g_{,n} = 0, \quad \text{for every } g \in \mathcal{A}, \quad (5.9)$$

where we recall that we have defined  $w_R^e \equiv w|_{\Omega \setminus \overline{D}}$  and  $w_R^i \equiv w|_{\overline{D}}$ .

When a cavity is present, then the transversal displacement in  $\Omega \setminus \overline{D}$  is given as the weak solution  $w_V \in H^2(\Omega \setminus \overline{D})$  to the boundary value problem

$$\begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w_V)) = 0, & \text{in } \Omega \setminus \overline{D}, \\ (\mathbb{P}\nabla^2 w_V)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \operatorname{div}(\mathbb{P}\nabla^2 w_V) \cdot n + ((\mathbb{P}\nabla^2 w_V)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega, \\ (\mathbb{P}\nabla^2 w_V)n \cdot n = 0, & \text{on } \partial D, \\ \operatorname{div}(\mathbb{P}\nabla^2 w_V) \cdot n + ((\mathbb{P}\nabla^2 w_V)n \cdot \tau)_{,s} = 0, & \text{on } \partial D. \end{cases} \quad \begin{matrix} (5.10) \\ (5.11) \\ (5.12) \\ (5.13) \\ (5.14) \end{matrix}$$

Finally, when the inclusion is absent, we shall denote by  $w_0 \in H^2(\Omega)$  the corresponding transversal displacement of the plate which will be given as the weak solution of the Neumann problem

$$\begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w_0)) = 0, & \text{in } \Omega, \\ (\mathbb{P}\nabla^2 w_0)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \operatorname{div}(\mathbb{P}\nabla^2 w_0) \cdot n + ((\mathbb{P}\nabla^2 w_0)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega. \end{cases} \quad \begin{matrix} (5.15) \\ (5.16) \\ (5.17) \end{matrix}$$

We shall denote by  $W_R$ ,  $W_V$ ,  $W_0$  the work exerted by the couple field  $\widehat{M}$  acting on  $\partial\Omega$  when  $D$  is a rigid inclusion, it is a cavity, or it is absent, respectively. By the weak formulation of the corresponding equilibrium problem it turns out that

$$W_R = - \int_{\partial\Omega} \widehat{M}_{\tau,s} w_R + \widehat{M}_n w_{R,n} = \int_{\Omega \setminus \overline{D}} \mathbb{P}\nabla^2 w_R \cdot \nabla^2 w_R, \quad (5.18)$$

$$W_V = - \int_{\partial\Omega} \widehat{M}_{\tau,s} w_V + \widehat{M}_n w_{V,n} = \int_{\Omega \setminus \overline{D}} \mathbb{P}\nabla^2 w_V \cdot \nabla^2 w_V, \quad (5.19)$$

$$W_0 = - \int_{\partial\Omega} \widehat{M}_{\tau,s} w_0 + \widehat{M}_n w_{0,n} = \int_{\Omega} \mathbb{P}\nabla^2 w_0 \cdot \nabla^2 w_0. \quad (5.20)$$

Note that the works  $W_R$ ,  $W_V$  and  $W_0$  are well defined since they are invariant with respect to the addition of any affine function to the displacement fields  $w_R$ ,  $w_V$  and  $w_0$ , respectively. In the following, the solutions  $w_V$  and  $w_0$  will be uniquely determined by imposing the normalization conditions

$$\int_{\partial D} w_V = 0, \quad \int_{\partial D} \nabla w_V = 0, \quad (5.21)$$

$$\int_{\Omega} w_0 = 0, \quad \int_{\Omega} \nabla w_0 = 0. \quad (5.22)$$

Concerning the solution  $w_R$ , we found convenient to normalize it by requiring that

$$w_R = 0, \quad \text{in } \overline{D}. \quad (5.23)$$

For a given positive number  $h_1$ , we denote by  $D_{h_1\rho_0}$  the set

$$D_{h_1\rho_0} = \{x \in D \mid \operatorname{dist}(x, \partial D) > h_1\rho_0\}. \quad (5.24)$$

We are now in position to state our size estimates. In the case of a rigid inclusion we have the following two theorems.

**Theorem 5.3.** *Let  $\Omega$  be a simply connected bounded domain in  $\mathbb{R}^2$ , such that  $\partial\Omega$  is of class  $C^{2,1}$  with constants  $\rho_0$ ,  $M_0$ , and satisfying (4.19). Let  $D$  be a simply connected open subset of  $\Omega$  with boundary  $\partial D$  of class  $C^{1,1}$ , satisfying (5.1), such that  $\Omega \setminus \overline{D}$  is connected and*

$$\text{area}(D_{h_1\rho_0}) \geq \frac{1}{2}\text{area}(D), \quad (5.25)$$

*for a given positive number  $h_1$ . Let the plate tensor  $\mathbb{P}$  given by (4.1) satisfy (4.2), (4.4), (4.30) and the dichotomy condition. Let  $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$  satisfy (4.26)–(4.28), with  $\Gamma$  satisfying (4.25). The following inequality holds*

$$\text{area}(D) \leq K\rho_0^2 \frac{W_0 - W_R}{W_0}, \quad (5.26)$$

*where the constant  $K > 0$  only depends on the quantities  $M_0$ ,  $M_1$ ,  $d_0$ ,  $h_1$ ,  $\gamma$ ,  $\delta_1$ ,  $M$ ,  $\delta_0$  and  $F$ .*

**Theorem 5.4.** *Let  $\Omega$  be a simply connected bounded domain in  $\mathbb{R}^2$ , such that  $\partial\Omega$  is of class  $C^{2,1}$  with constants  $\rho_0$ ,  $M_0$ , and satisfying (4.19). Let  $D$  be a simply connected domain satisfying (5.1), (5.2), such that  $\Omega \setminus \overline{D}$  is connected and the boundary  $\partial D$  is of class  $C^{3,1}$  with constants  $r\rho_0$ ,  $L$ . Let the plate tensor  $\mathbb{P}$  given by (4.1) satisfy (4.2), (4.4) and (4.30). Let  $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$  satisfy (4.27). The following inequality holds*

$$C\rho_0^2\Phi\left(\frac{W_0 - W_R}{W_0}\right) \leq \text{area}(D), \quad (5.27)$$

*where the function  $\Phi$  is given by*

$$[0, 1) \ni t \mapsto \Phi(t) = \frac{t^2}{1 - t}, \quad (5.28)$$

*and  $C > 0$  is a constant only depending on  $M_0$ ,  $M_1$ ,  $d_0$ ,  $L$ ,  $Q$ ,  $\gamma$  and  $M$ .*

When  $D$  is a cavity, the following bounds hold.

**Theorem 5.5.** *Let  $\Omega$  be a simply connected bounded domain in  $\mathbb{R}^2$ , such that  $\partial\Omega$  is of class  $C^{2,1}$  with constants  $\rho_0$ ,  $M_0$ , and satisfying (4.19). Let  $D$  be a simply connected open subset of  $\Omega$  with boundary  $\partial D$  of class  $C^{1,1}$ , satisfying (5.1), such that  $\Omega \setminus \overline{D}$  is connected and*

$$\text{area}(D_{h_1\rho_0}) \geq \frac{1}{2}\text{area}(D), \quad (5.29)$$

for a given positive number  $h_1$ . Let the plate tensor  $\mathbb{P}$  given by (4.1) satisfy (4.2), (4.4), (4.30) and the dichotomy condition. Let  $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$  satisfy (4.26)–(4.28), with  $\Gamma$  satisfying (4.25). The following inequality holds

$$\text{area}(D) \leq K \rho_0^2 \frac{W_V - W_0}{W_0}, \quad (5.30)$$

where the constant  $K > 0$  only depends on the quantities  $M_0, M_1, d_0, h_1, \gamma, \delta_1, M, \delta_0$  and  $F$ .

**Theorem 5.6.** *Let  $\Omega$  be a simply connected bounded domain in  $\mathbb{R}^2$ , such that  $\partial\Omega$  is of class  $C^{1,1}$  with constants  $\rho_0, M_0$ , and satisfying (4.19). Let  $D$  be a simply connected domain satisfying (5.1), (5.2), such that  $\Omega \setminus \overline{D}$  is connected and the boundary  $\partial D$  is of class  $C^{1,1}$  with constants  $r\rho_0, L$ . Let the plate tensor  $\mathbb{P}$  given by (4.1) satisfy (4.2) and (4.4), and such that  $\|\mathbb{P}\|_{C^{2,1}(\mathbb{R}^2)} \leq M'$ , where  $M'$  is a positive parameter. Let  $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$  satisfy (4.27). The following inequality holds*

$$C \rho_0^2 \Psi \left( \frac{W_V - W_0}{W_0} \right) \leq \text{area}(D), \quad (5.31)$$

where the function  $\Psi$  is given by

$$[0, +\infty) \ni t \mapsto \Psi(t) = \frac{t^2}{1+t}, \quad (5.32)$$

and  $C > 0$  is a constant only depending on  $M_0, M_1, d_0, L, Q, \gamma$  and  $M'$ .

## 5.2 Proof of Theorems 5.3 and 5.4

The starting point of the upper and lower estimates of the area of a rigid inclusion is the following energy estimate, in which the works  $W_0$  and  $W_R$  are compared.

**Lemma 5.7.** *Let  $\Omega$  be a simply connected bounded domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega$  of class  $C^{1,1}$ . Assume that  $D$  is a simply connected open set compactly contained in  $\Omega$ , with boundary  $\partial D$  of class  $C^{1,1}$  and such that  $\Omega \setminus \overline{D}$  is connected. Let the plate tensor  $\mathbb{P} \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$  given by (4.1) satisfy (4.2) and (4.4). Let  $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$  satisfy (4.27). Let  $w_R \in H^2(\Omega \setminus \overline{D})$ ,  $w_0 \in H^2(\Omega)$  be the solutions to problems (5.4)–(5.9) and (5.15)–(5.17), normalized as above. We have*

$$\int_D \mathbb{P} \nabla^2 w_0 \cdot \nabla^2 w_0 \leq W_0 - W_R = \int_{\partial D} M_n(w_R) w_{0,n} + V(w_R) w_0, \quad (5.33)$$

where we have denoted by

$$M_n(w_R) = -(\mathbb{P}\nabla^2 w_R)n \cdot n, \quad (5.34)$$

$$V(w_R) = \operatorname{div}(\mathbb{P}\nabla^2 w_R) \cdot n + ((\mathbb{P}\nabla^2 w_R)n \cdot \tau)_{,s} \quad (5.35)$$

the bending moment and the Kirchhoff shear on  $\partial D$  associated to  $w_R$ , respectively. Here,  $n$  denotes the exterior unit normal to  $\Omega \setminus \overline{D}$ .

*Proof.* The proof is based on the weak formulation of problems (5.4)–(5.9) and (5.15)–(5.17), and can be obtained by adapting the proof of the corresponding result for a rigid inclusion in an elastic body derived in ([M-R1], Lemma 3.1).  $\square$

*Proof of Theorem 5.3.* By (5.33) and from the strong convexity condition (4.4) we have

$$\int_D |\nabla^2 w_0|^2 \leq \gamma^{-1}(W_0 - W_R). \quad (5.36)$$

Estimate (5.26) can be obtained from the following lower bound for the elastic energy associated to  $w_0$  in  $D$

$$\int_D |\nabla^2 w_0|^2 \geq C \frac{\operatorname{area}(D)}{\rho_0^2} W_0, \quad (5.37)$$

where the constant  $C > 0$  only depends on  $M_0, M_1, d_0, h_1, \gamma, \delta_1, M, \delta_0$  and  $F$ . The above estimate was derived in ([M-R-Ve6], Theorem 3.1) for inclusions  $D$  satisfying the *fatness condition* (5.25) and its proof is based on a three sphere inequality for solutions to the plate equation (5.15) with anisotropic elastic coefficients obeying to the dichotomy condition.  $\square$

In order to prove Theorem 5.4 we need the following Poincaré inequalities of constructive type.

For a given positive number  $r > 0$ , we denote by  $D^{r\rho_0}$  the following set

$$D^{r\rho_0} = \{x \in \mathbb{R}^2 \mid 0 < \operatorname{dist}(x, D) < r\rho_0\}. \quad (5.38)$$

For  $D$  with Lipschitz boundary and  $u \in H^1(D)$  we define

$$u_D = \frac{1}{|D|} \int_D u, \quad u_{\partial D} = \frac{1}{|\partial D|} \int_{\partial D} u. \quad (5.39)$$

**Proposition 5.8.** *Let  $D$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , of Lipschitz class with constants  $r\rho_0, L$ , satisfying condition (5.2) with constant  $Q > 0$ . For every  $u \in H^1(D)$  we have*

$$\int_D |u - u_D|^2 \leq C_1 r^2 \rho_0^2 \int_D |\nabla u|^2, \quad (5.40)$$



$$\int_{\partial D} |u - u_{\partial D}|^2 \leq C_2 r \rho_0 \int_D |\nabla u|^2, \quad (5.41)$$

where  $C_1 > 0$ ,  $C_2 > 0$  only depend on  $L$  and  $Q$ .

If  $u \in H^1(D^{r\rho_0})$ , then

$$\int_{\partial D} |u - u_{\partial D}|^2 \leq C_3 r \rho_0 \int_{D^{r\rho_0}} |\nabla u|^2, \quad (5.42)$$

where  $C_3 > 0$  only depends on  $L$  and  $Q$ . Moreover, if  $u \in H^1(D^{r\rho_0})$  and  $u = 0$  on  $\partial D$ , then we have

$$\int_{D^{r\rho_0}} u^2 \leq C_4 r^2 \rho_0^2 \int_{D^{r\rho_0}} |\nabla u|^2, \quad (5.43)$$

where  $C_4 > 0$  only depends on  $L$  and  $Q$ .

*Proof.* We refer to [Al-M-R1] for a proof of the inequalities (5.40)–(5.42) and for a precise evaluation of the constants  $C_1$ ,  $C_2$ ,  $C_3$  in terms of the scale invariant bounds  $L$ ,  $Q$  regarding the regularity and shape of  $D$ .

Inequality (5.43) is a consequence of the following result. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with Lipschitz boundary with constants  $r\rho_0$ ,  $L$ , and such that  $\text{diam}(\Omega) \leq Qr\rho_0$ ,  $Q > 0$ . Let  $E$  be any measurable subset of  $\Omega$  with positive Lebesgue measure  $|E| > 0$ . For every  $v \in H^1(\Omega)$  we have

$$\int_{\Omega} |v - v_E|^2 \leq \left(1 + \sqrt{\frac{|\Omega|}{|E|}}\right)^2 C r^2 \rho_0^2 \int_{\Omega} |\nabla v|^2, \quad (5.44)$$

where the constant  $C > 0$  only depends on  $L$  and  $Q$ . The above inequality follows from Lemma 2.1 of [Al-M-R4] and from (5.40) applied to the function  $v$ , see also inequality (3.8) of [Al-M-R4]. Let us extend the function  $u \in H^1(D^{r\rho_0})$  into the interior of  $D$  by taking  $u \equiv 0$  in  $D$ , and let us continue to denote by  $u$  this extended function, with  $u \in H^1(D \cup D^{r\rho_0})$ . Inequality (5.43) follows from (5.44) by taking  $v = u$ ,  $\Omega = D \cup D^{r\rho_0}$  and  $E = D$ .  $\square$

*Proof of Theorem 5.4.* Let  $g = a + bx + cy$  be an affine function such that the function  $\tilde{w}_0 = w_0 + g$  satisfies

$$\int_{\partial D} \tilde{w}_0 = 0, \quad \int_{\partial D} \nabla \tilde{w}_0 = 0. \quad (5.45)$$

The function  $\tilde{w}_0 \in H^2(\Omega)$  is a solution of (5.15)–(5.17), and by (5.9), from

the right-hand side of (5.33) and by applying Hölder's inequality, we have

$$\begin{aligned} W_0 - W_R &\leq \left( \int_{\partial D} |M_n(w_R)|^2 \right)^{\frac{1}{2}} \left( \int_{\partial D} |\tilde{w}_{0,n}|^2 \right)^{\frac{1}{2}} + \\ &\quad + \left( \int_{\partial D} |V(w_R)|^2 \right)^{\frac{1}{2}} \left( \int_{\partial D} |\tilde{w}_0|^2 \right)^{\frac{1}{2}} = I_1 + I_2. \end{aligned} \quad (5.46)$$

We start by estimating  $I_1$ . By (5.41) and by the definition of  $\tilde{w}_0$  we have

$$\int_{\partial D} |\tilde{w}_{0,n}|^2 \leq Cr\rho_0 \int_D |\nabla^2 w_0|^2 \leq Cr\rho_0 \|\nabla^2 w_0\|_{L^\infty(D)}^2 \text{area}(D), \quad (5.47)$$

where the constant  $C > 0$  only depends on  $L$  and  $Q$ . By the Sobolev embedding theorem (see, for instance, [Ad]), by standard interior regularity estimates (see, for example, Theorem 8.3 in [M-R-Ve1]), by Proposition 5.8, by (4.4) and by (5.20) we have

$$\|\nabla^2 w_0\|_{L^\infty(D)} \leq \frac{C}{\rho_0^2} \|w_0\|_{H^2(\Omega)} \leq \frac{C}{\rho_0} \left( \int_{\Omega} |\nabla^2 w_0|^2 \right)^{\frac{1}{2}} \leq \frac{C}{\rho_0} W_0^{\frac{1}{2}}, \quad (5.48)$$

where the constant  $C > 0$  only depends on  $M_0$ ,  $M_1$ ,  $d_0$ ,  $\gamma$  and  $M$ .

The integral  $\int_{\partial D} |M_n(w_R)|^2$  can be estimated by using a trace inequality (see, for instance, [L-M]) and a  $H^3$ -regularity estimate up to the boundary  $\partial D$  for  $w_R$  (see, for example, Lemma 4.2 and Theorem 5.2 of [M-R-Ve5]):

$$\int_{\partial D} |M_n(w_R)|^2 \leq \frac{C}{r^5 \rho_0^5} \sum_{i=0}^3 (r\rho_0)^{2i} \int_{D^{\frac{r\rho_0}{2}}} |\nabla^i w_R|^2 \leq \frac{C}{r^5 \rho_0^5} \sum_{i=0}^2 (r\rho_0)^{2i} \int_{D^{r\rho_0}} |\nabla^i w_R|^2, \quad (5.49)$$

where the constant  $C > 0$  only depends on  $M_0$ ,  $M_1$ ,  $L$ ,  $Q$ ,  $\gamma$  and  $M$ . The function  $w_R$  belongs to  $H^2(D^{r\rho_0})$  and, by (5.23),  $w_R = 0$  and  $\nabla w_R = 0$  on  $\partial D$ . Therefore, by applying twice (5.43), by (4.4) and by (5.18) we have

$$\sum_{i=0}^2 (r\rho_0)^{2i} \int_{D^{r\rho_0}} |\nabla^i w_R|^2 \leq Cr^4 \rho_0^4 \int_{D^{r\rho_0}} |\nabla^2 w_R|^2 \leq Cr^4 \rho_0^4 W_R, \quad (5.50)$$

where the constant  $C > 0$  only depends on  $L$ ,  $Q$  and  $\gamma$ . By (5.47), (5.48), (5.49), (5.50) we have

$$I_1 \leq \frac{C}{\rho_0} W_0^{\frac{1}{2}} W_R^{\frac{1}{2}} (\text{area}(D))^{\frac{1}{2}}, \quad (5.51)$$

where the constant  $C > 0$  only depends on  $M_0$ ,  $M_1$ ,  $d_0$ ,  $L$ ,  $Q$ ,  $\gamma$  and  $M$ .

The control of the term  $I_2$  can be obtained similarly. By a standard Poincaré inequality, by (5.41) and by (5.48) we have

$$\begin{aligned} \int_{\partial D} |\tilde{w}_0|^2 &\leq Cr^2 \rho_0^2 \int_{\partial D} |\tilde{w}_{0,s}|^2 \leq Cr^3 \rho_0^3 \int_D |\nabla^2 w_0|^2 \leq \\ &\leq Cr^3 \rho_0^3 \|\nabla^2 w_0\|_{L^\infty(D)}^2 \text{area}(D) \leq Cr^3 \rho_0 W_0 \text{area}(D), \end{aligned} \quad (5.52)$$

where the constant  $C > 0$  only depends on  $M_0, M_1, d_0, L, Q, \gamma$  and  $M$ . Concerning the integral  $\int_{\partial D} |V(w_R)|^2$ , by using a trace inequality, a  $H^4$ -regularity estimate up to the boundary  $\partial D$  for  $w_R$  (see, for example, Lemma 4.3 and Theorem 5.3 in [M-R-Ve5]) and by (5.50), we have

$$\begin{aligned} \int_{\partial D} |V(w_R)|^2 &\leq \frac{C}{r^7 \rho_0^7} \sum_{i=0}^4 (r \rho_0)^{2i} \int_{D^{\frac{r \rho_0}{2}}} |\nabla^i w_R|^2 \leq \\ &\leq \frac{C}{r^3 \rho_0^3} \int_{D^{r \rho_0}} |\nabla^2 w_R|^2 \leq \frac{C}{r^3 \rho_0^3} W_R, \end{aligned} \quad (5.53)$$

where the constant  $C > 0$  only depends on  $M_0, M_1, L, Q, \gamma$  and  $M$ . Then, by (5.52) and (5.53) we have

$$I_2 \leq \frac{C}{\rho_0} W_0^{\frac{1}{2}} W_R^{\frac{1}{2}} (\text{area}(D))^{\frac{1}{2}}, \quad (5.54)$$

where the constant  $C > 0$  only depends on  $M_0, M_1, d_0, L, Q, \gamma$  and  $M$ . By (5.51) and (5.54), the inequality (5.27) follows.  $\square$

### 5.3 Proof of Theorems 5.5 and 5.6

As in the proof of the upper and lower estimates of the area of a rigid inclusion, we need to compare the works  $W_0$  and  $W_V$ . The analogue of Lemma 5.7 is the following result.

**Lemma 5.9.** *Let  $\Omega$  be a simply connected bounded domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega$  of class  $C^{1,1}$ . Assume that  $D$  is a simply connected open set compactly contained in  $\Omega$ , with boundary  $\partial D$  of class  $C^{1,1}$  and such that  $\Omega \setminus \overline{D}$  is connected. Let the plate tensor  $\mathbb{P} \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$  given by (4.1) satisfy (4.2) and (4.4). Let  $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$  satisfy (4.27). Let  $w_V \in H^2(\Omega \setminus \overline{D})$ ,  $w_0 \in H^2(\Omega)$  be the solutions to problems (5.10)–(5.14) and (5.15)–(5.17), normalized as above. We have*

$$\int_D \mathbb{P} \nabla^2 w_0 \cdot \nabla^2 w_0 \leq W_V - W_0 = \int_{\partial D} M_n(w_0) w_{V,n} + V(w_0) w_V, \quad (5.55)$$

where the functions  $M_n(w_0)$ ,  $V(w_0)$  are defined as in (5.34) and (5.35), and  $n$  denotes the exterior unit normal to  $\Omega \setminus \overline{D}$ .

*Proof.* As for Lemma 5.7, the proof is based on the weak formulation of problems (5.10)–(5.14) and (5.15)–(5.17), and can be obtained following the same guidelines of the corresponding result for a cavity in an elastic body derived in ([M-R1], Lemma 3.5). However, here we simplify the approach presented in [M-R1] without extending the function  $w_V$  in the interior of  $D$ .  $\square$

*Proof of Theorem 5.5.* The proof follows from the left-hand side of (5.55) by using the same arguments as in the proof of Theorem 5.3.  $\square$

*Proof of Theorem 5.6.* From the right-hand side of (5.55) and by applying Hölder's inequality, we have

$$\begin{aligned} W_V - W_0 &\leq \left( \int_{\partial D} |M_n(w_0)|^2 \right)^{\frac{1}{2}} \left( \int_{\partial D} |w_{V,n}|^2 \right)^{\frac{1}{2}} + \\ &\quad + \left( \int_{\partial D} |V(w_0)|^2 \right)^{\frac{1}{2}} \left( \int_{\partial D} |w_V|^2 \right)^{\frac{1}{2}} = J_1 + J_2. \end{aligned} \quad (5.56)$$

Let us estimate the integral  $J_2$ . By the Sobolev embedding theorem (see, for instance, [Ad]), by standard interior regularity estimates for  $w_0$  (see, for example, Lemma 1 in [M-R-Ve2]), by (5.40), by (4.4) and by (5.20) we have

$$\begin{aligned} \int_{\partial D} |V(w_0)|^2 &\leq C \|\nabla^3 w_0\|_{L^\infty(D)}^2 |\partial D| \leq \frac{C}{\rho_0^6} \|w_0\|_{H^2(\Omega)}^2 |\partial D| \leq \\ &\leq \frac{C}{\rho_0^4} \int_{\Omega} |\nabla^2 w_0|^2 |\partial D| \leq \frac{C}{\rho_0^4} W_0 |\partial D|, \end{aligned} \quad (5.57)$$

where the constant  $C > 0$  only depends on  $M_0$ ,  $M_1$ ,  $d_0$ ,  $\gamma$  and  $M'$ . By ([Al-R], Lemma 2.8) we have

$$|\partial D| \leq C \frac{\text{area}(D)}{r \rho_0}, \quad (5.58)$$

where the constant  $C > 0$  only depends on  $L$ .

To control the integral  $\int_{\partial D} |w_V|^2$  we use a standard Poincaré inequality on  $\partial D$ , Proposition 5.8, inequality (5.42), the strong convexity condition (4.4) for  $\mathbb{P}$  and the definition of  $W_V$ , that is

$$\begin{aligned} \int_{\partial D} |w_V|^2 &\leq C r^2 \rho_0^2 \int_{\partial D} |w_{V,s}|^2 \leq C r^3 \rho_0^3 \int_{D^{r\rho_0}} |\nabla^2 w_V|^2 \leq \\ &\leq C r^3 \rho_0^3 \int_{\Omega \setminus \overline{D}} \mathbb{P} \nabla^2 w_V \cdot \nabla^2 w_V = C r^3 \rho_0^3 W_V, \end{aligned} \quad (5.59)$$

where the constant  $C > 0$  only depends on  $L$ ,  $Q$  and  $\gamma$ . Then, by (5.57), (5.58) and (5.59), and since, trivially,  $r \leq K$ , where  $K > 0$  is a constant only depending on  $L$  and  $M_1$ , we have

$$J_2 \leq \frac{C}{\rho_0} W_0^{\frac{1}{2}} W_V^{\frac{1}{2}} (\text{area}(D))^{\frac{1}{2}}, \quad (5.60)$$

where the constant  $C > 0$  only depends on  $M_0$ ,  $M_1$ ,  $d_0$ ,  $L$ ,  $Q$ ,  $\gamma$  and  $M'$ . By using similar arguments we can also find the analogous bound for  $J_1$ , and the thesis follows.  $\square$

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